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Hilbert–Enskog–Chapman Expansion in the Turbulent Kinetic Theory of Gases. II

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This is a continuation of the article (I) with the same title. [Tian-quan Chen, preceding article *J. Stat. Phys.* **25**, 491 (1981).] We carry on with the Enskog–Chapman expansion here.

KEY WORDS: BBGGKY hierarchy; Enskog–Chapman expansion.

8.[3] 125 THIRD-ORDER CONSERVATION EQUATIONS

From 125 third-order summational invariants and the equation (3)[3] we can derive 125 third-order conservation equations. In order to write down these conservation equations we have to introduce the following notations:

$$S^{(l\quad,\ m\quad,\ n\quad)}_{i_1\cdots i_l, j_1\cdots j_m, k_1\cdots k_n}$$
$$= \int h(z,\hat{z},\check{z},t) \prod_{r=1}^{l} (v_{i_r} - u_{i_r}) \prod_{s=1}^{m} (\hat{v}_{j_s} - \hat{u}_{j_s}) \prod_{t=1}^{n} (\check{v}_{k_t} - \check{u}_{k_t})\, dv\, d\hat{v}\, d\check{v}$$

$$S^{(l+2\,,\ m\quad,\ n\quad)}_{i_1\cdots i_l, j_1\cdots j_m, k_1\cdots k_n} = S^{(l+2\,,\ m\quad,\ n\quad)}_{i_1\cdots i_l pp, j_1\cdots j_m, k_1\cdots k_n}$$

(on the right-hand side of the above equality, as well as throughout this section, the summation convention is implied), and

$$\frac{\Delta}{\Delta t} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} + \hat{u}_k \frac{\partial}{\partial \hat{x}_k} + \check{u}_k \frac{\partial}{\partial \check{x}_k}$$

The 125 third-order conservation equations can be divided into ten

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[3] The ordinal numbers of the sections and the equations in this paper are consecutive with those in the preceding one.[(1)] When we refer to an equation in Ref. 1, we shall just refer to the number of the equation, without citing Ref. 1.

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classes, of which the typical equations are as follows:

$$\frac{\Delta S^{(0,0,0)}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k} \right) S^{(0,0,0)} + \frac{\partial S^{(1,0,0)}_{k}}{\partial x_k} + \frac{\partial S^{(0,1,0)}_{k}}{\partial \hat{x}_k} + \frac{\partial S^{(0,0,1)}_{k}}{\partial \check{x}_k} = 0 \tag{62}$$

$$\frac{\Delta S^{(1,0,0)}_{i}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k} \right) S^{(1,0,0)}_{i} + \frac{Du_i}{Dt} S^{(0,0,0)} + S^{(1,0,0)}_{k} \frac{\partial u_i}{\partial x_k} + \frac{\partial S^{(1,1,0)}_{i,k}}{\partial \hat{x}_k} + \frac{\partial S^{(1,0,1)}_{i,\ k}}{\partial \check{x}_k} + \frac{\partial S^{(2,0,0)}_{ki}}{\partial x_k} = 0 \qquad (1 \leqslant i \leqslant 3) \tag{63}$$

$$\frac{\Delta S^{(1,1,0)}_{i,j}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k} \right) S^{(1,1,0)}_{i,j} + \frac{Du_i}{Dt} S^{(0,1,0)}_{j} + \frac{\hat{D}\hat{u}_j}{\hat{D}t} S^{(1,0,0)}_{i} + \frac{\partial u_i}{\partial x_k} S^{(1,1,0)}_{k,j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(1,1,0)}_{i,k} + \frac{\partial S^{(2,1,0)}_{ik,j}}{\partial x_k} + \frac{\partial S^{(1,2,0)}_{i,jk}}{\partial \hat{x}_k} + \frac{\partial S^{(1,1,1)}_{i,j,k}}{\partial \check{x}_k} = 0 \qquad (1 \leqslant i, j \leqslant 3) \tag{64}$$

$$\frac{\Delta S^{(1,1,1)}_{i,j,l}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k} \right) S^{(1,1,1)}_{i,j,l} + \frac{Du_i}{Dt} S^{(0,1,1)}_{j,l} + \frac{\hat{D}\hat{u}_j}{\hat{D}t} S^{(1,0,1)}_{i,\ l} + \frac{\check{D}\check{u}_l}{\check{D}t} S^{(1,1,0)}_{i,j} + \frac{\partial u_i}{\partial x_k} S^{(1,1,1)}_{k,j,l} + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(1,1,1)}_{i,k,l} + \frac{\partial \check{u}_l}{\partial \check{x}_k} S^{(1,1,1)}_{i,j,k} + \frac{\partial S^{(2,1,1)}_{ki,j,l}}{\partial x_k} + \frac{\partial S^{(1,2,1)}_{i,kj,l}}{\partial \hat{x}_k} + \frac{\partial S^{(1,1,2)}_{i,j,kl}}{\partial \check{x}_k} = 0 \qquad (1 \leqslant i, j, l \leqslant 3) \tag{65}$$

$$\frac{\Delta S^{(2,0,0)}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k} \right) S^{(2,0,0)} + 2 S^{(1,0,0)}_{k} \frac{Du_k}{Dt} + 2 \frac{\partial u_k}{\partial x_s} S^{(2,0,0)}_{ks} + \frac{\partial S^{(3,0,0)}_{k}}{\partial x_k} + \frac{\partial S^{(2,1,0)}_{k}}{\partial \hat{x}_k} + \frac{\partial S^{(2,0,1)}_{k}}{\partial \check{x}_k} = 0 \tag{66}$$

$$\begin{aligned}&\frac{\Delta S^{(2,1,0)}_{j}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,1,0)}_{j}+2S^{(1,1,0)}_{k,j}\frac{Du_k}{Dt}\\&\quad+S^{(2,0,0)}\frac{\hat{D}\hat{u}_j}{\hat{D}t}+2\frac{\partial u_k}{\partial x_s}S^{(2,1,0)}_{ks,j}+\frac{\partial \hat{u}_j}{\partial \hat{x}_k}S^{(2,1,0)}_{k}+\frac{\partial S^{(3,1,0)}_{k,j}}{\partial x_k}\\&\quad+\frac{\partial S^{(2,2,0)}_{kj}}{\partial \hat{x}_k}+\frac{\partial S^{(2,1,1)}_{j,k}}{\partial \check{x}_k}=0\qquad (1\leqslant j\leqslant 3)\end{aligned}\tag{67}$$

$$\begin{aligned}&\frac{\Delta S^{(2,1,1)}_{j,l}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,1,1)}_{j,l}+2\frac{Du_k}{Dt}S^{(1,1,1)}_{k,j,l}\\&\quad+\frac{\hat{D}\hat{u}_j}{\hat{D}t}S^{(2,0,1)}_{l}+\frac{\check{D}\check{u}_l}{\check{D}t}S^{(2,1,0)}_{j}+2\frac{\partial u_k}{\partial x_s}S^{(2,1,1)}_{ks,j,l}+\frac{\partial \hat{u}_j}{\partial \hat{x}_k}S^{(2,1,1)}_{k,l}\\&\quad+\frac{\partial \check{u}_l}{\partial \check{x}_k}S^{(2,1,1)}_{j,k}+\frac{\partial S^{(3,1,1)}_{k,j,l}}{\partial x_k}+\frac{\partial S^{(2,2,1)}_{kj,l}}{\partial \hat{x}_k}+\frac{\partial S^{(2,1,2)}_{j,kl}}{\partial \check{x}_k}=0\\&\qquad\qquad (1\leqslant j,l\leqslant 3)\end{aligned}\tag{68}$$

$$\begin{aligned}&\frac{\Delta S^{(2,2,1)}_{l}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,2,1)}_{l}+2\frac{Du_k}{Dt}S^{(1,2,1)}_{k,l}\\&\quad+2\frac{\hat{D}\hat{u}_k}{\hat{D}t}S^{(2,1,1)}_{k,l}+\frac{\check{D}\check{u}_l}{\check{D}t}S^{(2,2,0)}+2\frac{\partial u_k}{\partial x_s}S^{(2,2,1)}_{ks,l}+2\frac{\partial \hat{u}_k}{\partial \hat{x}_s}S^{(2,2,1)}_{ks,l}\\&\quad+\frac{\partial \check{u}_l}{\partial \check{x}_k}S^{(2,2,1)}_{k}+\frac{\partial S^{(3,2,1)}_{k\ l}}{\partial x_k}+\frac{\partial S^{(2,3,1)}_{k,l}}{\partial \hat{x}_k}+\frac{\partial S^{(2,2,2)}_{kl}}{\partial \check{x}_k}=0\\&\qquad\qquad (1\leqslant l\leqslant 3)\end{aligned}\tag{69}$$

$$\begin{aligned}&\frac{\Delta S^{(2,2,0)}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,2,0)}+2\frac{Du_k}{Dt}S^{(1,2,0)}_{k}\\&\quad+2\frac{\hat{D}\hat{u}_k}{\hat{D}t}S^{(2,1,0)}_{k}+2\frac{\partial u_k}{\partial x_s}S^{(2,2,0)}_{ks}+2\frac{\partial \hat{u}_k}{\partial \hat{x}_s}S^{(2,2,0)}_{ks}+\frac{\partial S^{(3,2,0)}_{k}}{\partial x_k}\\&\quad+\frac{\partial S^{(2,3,0)}_{k}}{\partial \hat{x}_k}+\frac{\partial S^{(2,2,1)}_{k}}{\partial \check{x}_k}=0\end{aligned}\tag{70}$$

$$\frac{\Delta S^{(2,2,2)}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,2,2)}+2\frac{Du_k}{Dt}S^{(1,2,2)}_{k}$$
$$+2\frac{\hat{D}\hat{u}_k}{\hat{D}t}S^{(2,1,2)}_{k}+2\frac{\check{D}\check{u}_k}{\check{D}t}S^{(2,2,1)}_{k}+2\frac{\partial u_k}{\partial x_s}S^{(2,2,2)}_{ks}+2\frac{\partial \hat{u}_k}{\partial \hat{x}_s}S^{(2,2,2)}_{ks}$$
$$+2\frac{\partial \check{u}_k}{\partial \check{x}_s}S^{(2,2,2)}_{ks}+\frac{\partial S^{(3,2,2)}_{k}}{\partial x_k}+\frac{\partial S^{(2,3,2)}_{k}}{\partial \hat{x}_k}+\frac{\partial S^{(2,2,3)}_{k}}{\partial \check{x}_k}=0 \tag{71}$$

9. THE SOLUTION OF EQUATION (30)

In order to solve equation (30) we have to calculate the right-hand side of the equation. On account of (34), (45), (46), and (47), we have

$$\mathcal{J}[g^{(1)}g^{(1)}]_{\hat{z}\check{z}}$$
$$=\int\big[g^{(1)}(z',\hat{z})g^{(1)}(\tilde{z}',\check{z})+g^{(1)}(\tilde{z}',\hat{z})g^{(1)}(z',\check{z})$$
$$-g^{(1)}(z,\hat{z})g^{(1)}(\tilde{z},\check{z})-g^{(1)}(\tilde{z},\hat{z})g^{(1)}(z,\check{z})\big]B(\theta)\,d\epsilon\,d\theta\,d\tilde{v}$$
$$=f^{(0)}\hat{f}^{(0)}\check{f}^{(0)}\Bigg[\sum_{0\leqslant i,j,s,t\leqslant 4}a_{ij}(x,\hat{x})a_{st}(x,\check{x})\hat{\psi}_j\check{\psi}_t$$
$$\times\int\tilde{f}^{(0)}(\psi_i'\tilde{\psi}_s'+\tilde{\psi}_i'\psi_s'-\psi_i\tilde{\psi}_s-\tilde{\psi}_i\psi_s)B(\theta)\,d\epsilon\,d\theta\,d\tilde{v}\Bigg]$$
$$=f^{(0)}\hat{f}^{(0)}\check{f}^{(0)}\sum_{0\leqslant j,t\leqslant 4}\hat{\psi}_j\check{\psi}_t\Bigg\{6B_1n\sum_{1\leqslant i,s\leqslant 3}a_{ij}(x,\hat{x})a_{st}(x,\check{x})\left(w_iw_s-\frac{1}{3}\delta_{is}|w|^2\right)$$
$$+4B_1n\sum_{1\leqslant i\leqslant 3}\big[a_{ij}(x,\hat{x})a_{4t}(x,\check{x})+a_{4j}(x,\hat{x})$$
$$\times a_{it}(x,\check{x})\big]\left(|w|^2w_i-\frac{5KT}{m}w_i\right)$$
$$+4B_1na_{4j}(x,\hat{x})a_{4t}(x,\check{x})$$
$$\times\left[|w|^4-10|w|^2\frac{KT}{m}+15\left(\frac{KT}{m}\right)^2\right]\Bigg\} \tag{72}$$

The expressions for $\mathcal{J}[g^{(1)}g^{(1)}]_{z\hat{z}}$ and $\mathcal{J}[g^{(1)}g^{(1)}]_{z\check{z}}$ can be obtained from the above expression by exchanging $z\leftrightarrow\check{z}$ and $z\leftrightarrow\hat{z}$, respectively. Multiplying the sum of these three expressions by (-1), we get the expression for the right-hand side of equation (30). In virtue of Wang-Chang and Uhlenbeck's results on the eigenfunctions and eigenvalues of the linear Boltzmann operator,[2] we get the solution of equation (30) as follows:

$$h^{(2)} = f^{(0)}\hat{f}^{(0)}\check{f}^{(0)} \sum_{0\leqslant j,t\leqslant 4} \hat{\psi}_j\check{\psi}_t \Bigg\{ \sum_{1\leqslant i,s\leqslant 3} a_{ij}(x,\hat{x})a_{st}(x,\check{x})\Big(w_iw_s - \frac{1}{3}\delta_{is}|w|^2\Big)$$

$$+ \sum_{1\leqslant i\leqslant 3} \big[a_{ij}(x,\hat{x})a_{4t}(x,\check{x}) + a_{4j}(x,\hat{x})a_{it}(x,\check{x})\big]$$

$$\times\Big(|w|^2w_i - \frac{5KT}{m}w_i\Big) + a_{4j}(x,\hat{x})a_{4t}(x,\check{x})$$

$$\times\left[|w|^4 - 10|w|^2\frac{KT}{m} + 15\Big(\frac{KT}{m}\Big)^2\right]\Bigg\}$$

+ (the sum of two expressions obtained from the above expression by exchanging $z\leftrightarrow$ and $z\leftrightarrow\hat{z}$, respectively)

$$+f^{(0)}\hat{f}^{(0)}\check{f}^{(0)} \sum_{0\leqslant i,j,k\leqslant 4} b_{ij}\psi_i\hat{\psi}_j\check{\psi}_k$$

On account of the additional integral conditions imposed on $h^{(2)}$ we have the following relations between 125 S's and 125 b_{ijk}'s. 125 b_{ijk}'s can be divided into ten classes each of which is expressed in terms of S's as follows (the indices i, j, k in the following formulas denote the numbers 1, 2, and 3. The numbers 0 and 4 will be explicitly indicated):

$$b_{ijk} = \frac{m^3}{n\hat{n}\check{n}K^3T\hat{T}\check{T}} S^{(1,1,1)}_{i,j,k} \tag{73}$$

$$b_{0jk} = \frac{m^3}{2n\hat{n}\check{n}K^3T\hat{T}\check{T}} \left(\frac{5KT}{m} S^{(0,1,1)}_{j,k} - S^{(2,1,1)}_{j,k}\right) \tag{74}$$

$$b_{4jk} = \frac{m^4}{6n\hat{n}\check{n}K^4T^2\hat{T}\check{T}} \left(S^{(2,1,1)}_{j,k} - \frac{3KT}{m} S^{(0,1,1)}_{j,k}\right) \tag{75}$$

$$b_{44k} = \frac{m^5}{36n\hat{n}\check{n}K^5T^2\hat{T}^2\check{T}} \left(S^{(2,2,1)}_{k} - \frac{3K\hat{T}}{m} S^{(2,0,1)}_{k} - \frac{3KT}{m} S^{(0,2,1)}_{k} + \frac{9K^2T\hat{T}}{m^2} S^{(0,0,1)}_{k}\right) \tag{76}$$

$$b_{04k} = \frac{m^4}{12n\hat{n}\check{n}K^4T\hat{T}^2\check{T}} \left(- S^{(2,2,1)}_{k} + \frac{3K\hat{T}}{m} S^{(2,0,1)}_{k} + \frac{5KT}{m} S^{(0,2,1)}_{k} - \frac{15K^2T\hat{T}}{m^2} S^{(0,0,1)}_{k}\right) \tag{77}$$

$$b_{00k} = \frac{m^3}{4n\hat{n}\check{n}K^3T\hat{T}\check{T}}\left(S^{(2,2,\underset{k}{1})} - \frac{5K\hat{T}}{m}S^{(2,0,\underset{k}{1})} - \frac{5KT}{m}S^{(0,2,\underset{k}{1})} + \frac{25K^2T\hat{T}}{m^2}S^{(0,0,\underset{k}{1})}\right) \tag{78}$$

$$b_{444} = \frac{m^6}{216n\hat{n}\check{n}K^6T^2\hat{T}^2\check{T}^2}\left[S^{(2,2,2)} - \frac{3K}{m}(\check{T}S^{(2,2,0)} + \hat{T}S^{(2,0,2)} + TS^{(0,2,2)}) + \frac{9K^2}{m^2}(T\hat{T}S^{(0,0,2)} + T\check{T}S^{(0,2,0)} + \hat{T}\check{T}S^{(2,0,0)}) - 27\frac{K^3T\hat{T}\check{T}}{m^3}S^{(0,0,0)}\right] \tag{79}$$

$$b_{404} = \frac{m^5}{72n\hat{n}\check{n}K^5T^2\hat{T}\check{T}^2}\left[-S^{(2,2,2)} + 3\left(\frac{K\check{T}}{m}S^{(2,2,0)} + \frac{KT}{m}S^{(0,2,2)}\right) + 5\frac{K\hat{T}}{m}S^{(2,0,2)} - 9\frac{K^2T\check{T}}{m^2}S^{(0,2,0)} - 15\left(\frac{K^2T\hat{T}}{m^2}S^{(0,0,2)} + \frac{K^2\hat{T}\check{T}}{m^2}S^{(2,0,0)}\right) + 45\frac{K^3T\hat{T}\check{T}}{m^3}S^{(0,0,0)}\right] \tag{80}$$

$$b_{004} = \frac{m^4}{24n\hat{n}\check{n}K^4T\hat{T}\check{T}^2}\left[S^{(2,2,2)} - 5\left(\frac{K\hat{T}}{m}S^{(2,0,2)} + \frac{KT}{m}S^{(0,2,2)}\right) - 3\frac{K\check{T}}{m}S^{(2,2,0)} + 15\left(\frac{K^2\hat{T}\check{T}}{m^2}S^{(2,0,0)} + \frac{K^2T\check{T}}{m^2}S^{(0,2,0)}\right) + 25\frac{K^2T\hat{T}}{m^2}S^{(0,0,2)} - 75\frac{K^3T\hat{T}\check{T}}{m^3}S^{(0,0,0)}\right] \tag{81}$$

$$b_{000} = \frac{m^3}{8n\hat{n}\check{n}K^3T\hat{T}\check{T}}\left[-S^{(2,2,2)} + 5\left(\frac{KT}{m}S^{(0,2,2)} + \frac{K\hat{T}}{m}S^{(2,0,2)} + \frac{K\check{T}}{m}S^{(2,2,0)}\right) - 25\left(\frac{K^2\hat{T}\check{T}}{m^2}S^{(2,0,0)} + \frac{K^2T\check{T}}{m^2}S^{(0,2,0)} + \frac{K^2T\hat{T}}{m^2}S^{(0,0,2)}\right) + 125\frac{K^3T\hat{T}\check{T}}{m^3}S^{(0,0,0)}\right] \tag{82}$$

10. THE CALCULATION OF $J[h^{(2)}(z',\hat{z},\tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})]_{\tilde{x}=x}$

It is easily seen from (72) that

$$
\begin{aligned}
&J\Big[h^{(2)}(z',\hat{z},\tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})\Big]_{\tilde{x}=x} \\
&= \hat{f}^{(0)} \sum_{0\leqslant j\leqslant 4} \hat{\psi}_j \Bigg\{ \sum_{\substack{0\leqslant t\leqslant 4\\ 1\leqslant i,s\leqslant 3}} a_{ij}(x,\hat{x}) a_{st}(x,x) \\
&\qquad \times J\Big[f^{(0)}\tilde{f}^{(0)} \Big\{ d_c \tilde{\psi}_t \Big(w_i w_s - \frac{1}{3}\delta_{is}|w|^2 \Big) \Big\} \Big] \\
&\qquad + \sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant t\leqslant 4}} \big[a_{ij}(x,\hat{x}) a_{4t}(x,x) + a_{4j}(x,\hat{x}) a_{it}(x,x) \big] \\
&\qquad \times J\Big[f^{(0)}\tilde{f}^{(0)} \Big\{ d_c \tilde{\psi}_t \Big(|w|^2 w_i - \frac{5KT}{m} w_i \Big) \Big\} \Big] \\
&\qquad + \sum_{0\leqslant t\leqslant 4} a_{4j}(x,\hat{x}) a_{4t}(x,x) \\
&\qquad \times J\Big[f^{(0)}\tilde{f}^{(0)} \Big\{ d_c \tilde{\psi}_t \Big(|w|^4 - 10|w|^2 \frac{KT}{m} + 15\Big(\frac{KT}{m}\Big)^2 \Big) \Big\} \Big] \Bigg\} \\
&+ \frac{1}{2} \hat{f}^{(0)} \sum_{0\leqslant j,t\leqslant 4} J\big[f^{(0)}\tilde{f}^{(0)} (d_c \psi_j \tilde{\psi}_t) \big] \\
&\times \Bigg\{ \sum_{1\leqslant i,s\leqslant 3} a_{ij}(\hat{x},x) a_{st}(\hat{x},x) \Big(\hat{w}_i \hat{w}_s - \frac{1}{3}\delta_{is}|\hat{w}|^2 \Big) \\
&\quad + \sum_{1\leqslant i\leqslant 3} \big[a_{ij}(\hat{x},x) a_{4t}(\hat{x},x) + a_{4j}(\hat{x},x) a_{it}(\hat{x},x) \big] \Big(|\hat{w}|^2 \hat{w}_i - \frac{5K\hat{T}}{m} \hat{w}_i \Big) \\
&\quad + a_{4j}(\hat{x},x) a_{4t}(\hat{x},x) \Big[|\hat{w}|^4 - 10|\hat{w}|^2 \frac{K\hat{T}}{m} + 15\Big(\frac{K\hat{T}}{m}\Big)^2 \Big] \\
&\quad + \sum_{0\leqslant i\leqslant 4} b_{jit}(x,\hat{x},x) \hat{\psi}_i \Bigg\}
\end{aligned}
\tag{83}
$$

Where d_c denotes the collision difference operator:

$$d_c\tilde{f}g = \tilde{f}'g' + f'\tilde{g}' - \tilde{f}g - f\tilde{g}$$

In order to calculate the right-hand side of the equation (83), besides the formulas (45), (46), (47) and Wang-Chang and Uhlenbeck's results on the eigenfunctions and eigenvalues of the linear Boltzmann integral operator we need some additional formulas.

Let A, B, and C be three (three-dimensional) vectors with coordinates a_i, b_i, and c_i ($i = 1,2,3$), respectively. The triadic product ABC of the vectors A, B, and C is defined as a tensor of rank 3 with components

$$(ABC)_{ijk} = a_i b_j c_k$$

Moreover, δA, $A\delta$ and $\delta^{1/2}A\delta^{1/2}$ denote three tensors of rank 3 with the following components, respectively:

$$(\delta A)_{ijk} = \delta_{ij}a_k, \qquad (A\delta)_{ijk} = a_i\delta_{jk}, \qquad (\delta^{1/2}A\delta^{1/2})_{ijk} = \delta_{ik}a_j$$

Now we begin to derive the formulas we need. A simple calculation shows that

$$\begin{aligned}
&\tilde{w}'w'w' + w'\tilde{w}'\tilde{w}' - \tilde{w}ww - w\tilde{w}\tilde{w}\\
&\quad= [\tilde{w} - \alpha(\alpha\cdot U)][w + \alpha(\alpha\cdot U)][w + \alpha(\alpha\cdot U)]\\
&\qquad + [w + \alpha(\alpha\cdot U)][\tilde{w} - \alpha(\alpha\cdot U)][\tilde{w} - \alpha(\alpha\cdot U)] - \tilde{w}ww - w\tilde{w}\tilde{w}\\
&\quad= (\tilde{w}w - w\tilde{w})\alpha(\alpha\cdot U) + (\tilde{w}\alpha w - w\alpha\tilde{w})(\alpha\cdot U) + (\tilde{w} + w)\alpha\alpha(\alpha\cdot U)^2\\
&\qquad + \alpha(\tilde{w}\tilde{w} - ww)(\alpha\cdot U) - \alpha(w + \tilde{w})\alpha(\alpha\cdot U)^2 - \alpha\alpha(w + \tilde{w})(\alpha\cdot U)^2
\end{aligned}$$

The following formula is similar to formula (44) and can be proved in a similar way:

$$\int \alpha A\alpha\, d\epsilon = \pi\sin^2\theta\,\delta^{1/2}A\delta^{1/2} + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{UAU}{|U|^2} \tag{84}$$

From formulas (43), (44), (84) and the lemma in Section 6 we can derive the following formula:

$$\begin{aligned}
\int d_c(\tilde{w}ww)\, d\epsilon &= (\tilde{w}w - w\tilde{w})U\frac{2\pi\cos\theta}{|U|}(\alpha\cdot U) + (\tilde{w}Uw - wU\tilde{w})2\pi\cos^2\theta\\
&\quad + (\alpha\cdot U)^2(\tilde{w} + w)\left[\pi\sin^2\theta\delta + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{UU}{|U|^2}\right]\\
&\quad + U(\tilde{w}\tilde{w} - ww)2\pi\cos^2\theta\\
&\quad - (\alpha\cdot U)^2\left[\pi\sin^2\theta\delta^{1/2}(w + \tilde{w})\delta^{1/2}\right.\\
&\qquad\qquad \left. + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{U(w + \tilde{w})U}{|U|^2}\right]\\
&\quad - (\alpha\cdot U)^2\left[\pi\sin^2\theta\delta + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{UU}{|U|^2}\right](w + \tilde{w}) =
\end{aligned}$$

$$= \pi \cos^2\theta \sin^2\theta \Big[(\tilde{w}+w)\delta |U|^2 - \delta^{1/2}(\tilde{w}+w)\delta^{1/2}|U|^2$$

$$- \delta(\tilde{w}+w)|U|^2 - 3(\tilde{w}+w)UU$$

$$+ 3U(\tilde{w}+w)U + 3UU(\tilde{w}+w) \Big]$$

$$= \pi \cos^2\theta \sin^2\theta \Big\{ 3(\tilde{w}w\tilde{w} + w\tilde{w}w + \tilde{w}\tilde{w}\tilde{w} + \tilde{w}\tilde{w}w + ww\tilde{w} + www)$$

$$- 9(w\tilde{w}\tilde{w} + \tilde{w}ww) + (|w|^2 + |\tilde{w}|^2 - 2w\cdot\tilde{w})$$

$$\times \Big[(\tilde{w}+w)\delta - \delta^{1/2}(\tilde{w}+w)\delta^{1/2} - \delta(\tilde{w}+w) \Big] \Big\}$$

Hence,

$$\int f^{(0)}\tilde{f}^{(0)} d_c(\tilde{w}ww) B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= B_1 n f^{(0)} \Big[\frac{3KT}{m} \delta^{1/2} w \delta^{1/2} + \frac{3KT}{m} \delta w + 3www - \frac{9KT}{m} w\delta$$

$$+ w\delta\Big(|w|^2 + \frac{3KT}{m}\Big) - \frac{2KT}{m} w\delta - \Big(|w|^2 + \frac{3KT}{m}\Big)\delta^{1/2} w \delta^{1/2}$$

$$+ \frac{2KT}{m}\delta^{1/2} w \delta^{1/2} - \Big(|w|^2 + \frac{3KT}{m}\Big)\delta w + \frac{2KT}{m}\delta w \Big]$$

$$= B_1 n f^{(0)} \Big[|w|^2 (w\delta - \delta w - \delta^{1/2} w \delta^{1/2})$$

$$+ \frac{2KT}{m}(\delta^{1/2} w \delta^{1/2} + \delta w - 4w\delta) + 3www \Big] \tag{85}$$

Combining (85) with (46), we have

$$\int f^{(0)}\tilde{f}^{(0)} d_c \Big[\tilde{w}\Big(ww - \frac{1}{3}|w|^2\delta \Big)\Big] B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= B_1 n f^{(0)} \Big\{ 3\Big[www - \frac{1}{5}(w\delta + \delta w + \delta^{1/2} w \delta^{1/2})|w|^2 \Big]$$

$$+ \frac{4}{15}\Big(|w|^2 - \frac{5KT}{m}\Big) w\delta - \frac{2}{5}\Big(|w|^2 - \frac{5KT}{m}\Big)$$

$$\times (\delta w + \delta^{1/2} w \delta^{1/2}) \Big\}$$

$$|\tilde{w}'|^2 w'w' + |w'|^2 \tilde{w}'\tilde{w}' - |\tilde{w}|^2 ww - |w|^2 \tilde{w}\tilde{w}$$
$$= \left[|\tilde{w}|^2 + (\alpha \cdot U)^2 - 2(\alpha \cdot U)(\alpha \cdot \tilde{w})\right]\left[w + \alpha(\alpha \cdot U)\right]\left[w + \alpha(\alpha \cdot U)\right]$$
$$+ \left[|w|^2 + (\alpha \cdot U)^2 + 2(\alpha \cdot U)(\alpha \cdot w)\right]\left[\tilde{w} - \alpha(\alpha \cdot U)\right]\left[\tilde{w} - \alpha(\alpha \cdot U)\right]$$
$$- |\tilde{w}|^2 ww - |w|^2 \tilde{w}\tilde{w} = |\tilde{w}|^2(\alpha \cdot U)(\alpha w + w\alpha) - |w|^2(\alpha \cdot U)(\alpha\tilde{w} + \tilde{w}\alpha)$$
$$+ (|w|^2 + |\tilde{w}|^2)(\alpha \cdot U)^2 \alpha\alpha + (ww + \tilde{w}\tilde{w})(\alpha \cdot U)^2 - (\alpha U + U\alpha)(\alpha \cdot U)^3$$
$$- 2(\alpha \cdot U)\left[(\alpha \cdot \tilde{w})ww - (\alpha \cdot w)\tilde{w}\tilde{w}\right]$$
$$- 2(\alpha \cdot U)^2\left[(\alpha \cdot \tilde{w})(w\alpha + \alpha w) + (\alpha \cdot w)(\tilde{w}\alpha + \alpha\tilde{w})\right] \tag{86}$$

The following formula can be derived in a way similar to that of the formula (44):

$$\int (\alpha \cdot A) B\alpha \, d\epsilon = \pi \sin^2\theta BA + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{(A \cdot U)BU}{|U|^2} \tag{87}$$

From formulas (43), (44), (87) and the lemma in Section 6, we can derive the following formula:

$$\int d_c(|w|^2 \tilde{w}\tilde{w})\, d\epsilon$$
$$= \pi \cos^2\theta \sin^2\theta\left[(|\tilde{w}|^2 + |w|^2)(|U|^2\delta - 3UU) + 4|U|^2 UU\right.$$
$$- 4|U|^2(w\tilde{w} + \tilde{w}w) + 6(U \cdot \tilde{w})(wU + Uw)$$
$$\left. + 6(U \cdot w)(\tilde{w}U + U\tilde{w})\right]$$
$$= \pi \cos^2\theta \sin^2\theta \left\{|w|^2 ww + |\tilde{w}|^2 \tilde{w}\tilde{w} + (|w|^2 + |\tilde{w}|^2)(\tilde{w}w + w\tilde{w})\right.$$
$$+ 4(w \cdot \tilde{w})(ww + \tilde{w}\tilde{w} + \tilde{w}w + w\tilde{w}) + (|\tilde{w}|^2 + |w|^2)$$
$$\left.\left[|\tilde{w}|^2 + |w|^2 - 2(\tilde{w} \cdot w)\right]\delta - 11(|w|^2 \tilde{w}\tilde{w} + |\tilde{w}|^2 ww)\right\}$$

Hence,

$$\int f^{(0)}\tilde{f}^{(0)} d_c(|w|^2 \tilde{w}\tilde{w}) B(\theta)\, d\theta\, d\epsilon\, d\tilde{v} = B_1 n f^{(0)}\left[|w|^4\delta + |w|^2 ww - 25\frac{KT}{m} ww\right.$$
$$\left. - 5\frac{KT}{m}|w|^2\delta + 20\left(\frac{KT}{m}\right)^2\delta\right] \tag{88}$$

By virtue of the equalities (88), (47) and Uhlenbeck and Wang-Chang's result about the eigenfunctions and eigenvalues of the linear Boltzmann integral operator, we get

$$\int f^{(0)}\tilde{f}^{(0)}d_c\left[\left(|w|^2 - \frac{3KT}{m}\right)\left(\tilde{w}\tilde{w} - \frac{1}{3}|\tilde{w}|^2\delta\right)\right]B(\theta)\,d\theta\,d\epsilon\,d\tilde{v}$$

$$= -\frac{1}{3}B_1 n f^{(0)}\left(|w|^4\delta - 3|w|^2 ww - \frac{7KT}{m}|w|^2\delta + 21\frac{KT}{m}ww\right) \quad (89)$$

$$|w'|^2 w'\tilde{w}' + |\tilde{w}'|^2\tilde{w}'w' - |w|^2 w\tilde{w} - |\tilde{w}|^2\tilde{w}w$$
$$= |w + \alpha(\alpha\cdot U)|^2\left[w + \alpha(\alpha\cdot U)\right]\left[\tilde{w} - \alpha(\alpha\cdot U)\right]$$
$$+ |\tilde{w} - \alpha(\alpha\cdot U)|^2\left[\tilde{w} - \alpha(\alpha\cdot U)\right]\left[w + \alpha(\alpha\cdot U)\right]$$
$$- |w|^2 w\tilde{w} - |\tilde{w}|^2\tilde{w}w$$
$$= |w|^2(\alpha\cdot U)(\alpha\tilde{w} - w\alpha) + |\tilde{w}|^2(\alpha\cdot U)(\tilde{w}\alpha - \alpha w)$$
$$- (|w|^2 + |\tilde{w}|^2)(\alpha\cdot U)\alpha\alpha + 2(\alpha\cdot U)\left[(\alpha\cdot w)w\tilde{w} - (\alpha\cdot\tilde{w})\tilde{w}w\right]$$
$$+ 2(\alpha\cdot U)^2\left[(\alpha\cdot w)(\alpha\tilde{w} - w\alpha) - (\alpha\cdot\tilde{w})(\tilde{w}\alpha - \alpha w)\right]$$
$$+ (\alpha\cdot U)^2(w\tilde{w} + \tilde{w}w) + (\alpha\cdot U)^3(\alpha\tilde{w} + \tilde{w}\alpha - \alpha w - w\alpha)$$

From formulas (43), (44), (87) and the lemma in Section 6 we can derive the following formula:

$$\int d_c(|w|^2 w\tilde{w})\,d\epsilon$$
$$= -(|w|^2 + |\tilde{w}|^2)\pi\cos^2\theta\sin^2\theta(|U|^2\delta - 3UU)$$
$$- 2\pi\cos^2\theta\sin^2\theta|U|^2(U\tilde{w} + \tilde{w}U - Uw - wU)$$
$$+ 2\pi\cos^2\theta\sin^2\theta\big[|U|^2 w\tilde{w} - 3(U\cdot w)U\tilde{w} - |U|^2 ww$$
$$+ 3(U\cdot w)wU + |U|^2\tilde{w}w - 3(U\cdot\tilde{w})Uw$$
$$- |U|^2\tilde{w}\tilde{w} + 3(U\cdot\tilde{w})\tilde{w}U\big]$$
$$= \pi\cos^2\theta\sin^2\theta\big\{3(|w|^2 + |\tilde{w}|^2)(ww + \tilde{w}\tilde{w}) + 3|w|^2\tilde{w}w$$
$$+ 3|\tilde{w}|^2 w\tilde{w} - 9|w|^2 w\tilde{w} - 9|\tilde{w}|^2\tilde{w}w$$
$$- (|w|^2 + |\tilde{w}|^2)\left[|w|^2 + |\tilde{w}|^2 - 2(w\cdot\tilde{w})\right]\delta\big\}$$

Hence,

$$\int f^{(0)}\tilde{f}^{(0)} d_c(|w|^2 w\tilde{w}) B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= B_1 n f^{(0)}\left(-|w|^4\delta + 3ww|w|^2 - \frac{3KT}{m}|w|^2\delta + \frac{9KT}{m} ww\right) \quad (90)$$

Combining formula (90) with formula (45), we have

$$\int f^{(0)}\tilde{f}^{(0)} d_c\left[\left(|w|^2 w - \frac{5KT}{m} w\right)\tilde{w}\right] B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= B_1 n f^{(0)}\left(-|w|^4\delta + 3|w|^2 ww + \frac{7KT}{m}|w|^2\delta - \frac{21KT}{m} ww\right) \quad (91)$$

$$|w'|^2|\tilde{w}'|^2\tilde{w}' + |\tilde{w}'|^2|w'|^2 w' - |w|^2|\tilde{w}|^2\tilde{w} - |\tilde{w}|^2|w|^2 w$$

$$= |w + \alpha(\alpha\cdot U)|^2|\tilde{w} - \alpha(\alpha\cdot U)|^2\left[\tilde{w} - \alpha(\alpha\cdot U)\right] + |\tilde{w} - \alpha(\alpha\cdot U)|^2$$

$$\times |w + \alpha(\alpha\cdot U)|^2\left[w + \alpha(\alpha\cdot U)\right] - |w|^2|\tilde{w}|^2(w + \tilde{w})$$

$$= (w + \tilde{w})\Big\{(|w|^2 + |\tilde{w}|^2)(\alpha\cdot U)^2 + 2(\alpha\cdot U)$$

$$\times\left[(\alpha\cdot w)|\tilde{w}|^2 - (\alpha\cdot\tilde{w})|w|^2\right]$$

$$-(\alpha\cdot U)^4 - 4(\alpha\cdot w)(\alpha\cdot\tilde{w})(\alpha\cdot U)^2\Big\}$$

On account of formulas (43), (44) and the lemma in Section 6 we have

$$\int d_c(|w|^2|\tilde{w}|^2\tilde{w})\, d\epsilon$$

$$= 2\pi\cos^2\theta\sin^2\theta\left[|U|^4 + 6(U\cdot w)(U\cdot\tilde{w}) - 2|U|^2(w\cdot\tilde{w})\right](w + \tilde{w})$$

$$= 2\pi\cos^2\theta\sin^2\theta\left[2(w\cdot\tilde{w})^2 - 4|w|^2|\tilde{w}|^2 + |w|^4 + |\tilde{w}|^4\right](w + \tilde{w})$$

Hence,

$$\int f^{(0)}\tilde{f}^{(0)} d_c(|w|^2|\tilde{w}|^2\tilde{w}) B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= 2B_1 n f^{(0)}\left[|w|^4 w - 10\frac{KT}{m}|w|^2 w + 15\left(\frac{KT}{m}\right)^2 w\right] \quad (92)$$

Combining formula (92) with formula (46) and Uhlenbeck and Wang-Chang's result about the eigenfunctions and eigenvalues of the linear Boltzmann integral operator, we have

$$\int f^{(0)}\tilde{f}^{(0)} d_c\left[\left(|w|^2 w - \frac{5KT}{m} w\right)\left(|\tilde{w}|^2 - \frac{3K\tilde{T}}{m}\right)\right] B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

$$= 2B_1 n f^{(0)}\left[|w|^4 w - \frac{14KT}{m}|w|^2 w + 35\left(\frac{KT}{m}\right)^2 w\right] \quad (93)$$

$$
\begin{aligned}
&|\tilde{w}'|^4 w' + |w'|^4 \tilde{w}' - |\tilde{w}|^4 w - |w|^4 \tilde{w} \\
&\quad = |\tilde{w} - \alpha(\alpha \cdot U)|^4 \big[w + \alpha(\alpha \cdot U) \big] \\
&\qquad + |w + \alpha(\alpha \cdot U)|^4 \big[\tilde{w} - \alpha(\alpha \cdot U) \big] - |\tilde{w}|^4 w - |w|^4 \tilde{w} \\
&\quad = (\alpha \cdot U)^4 (w + \tilde{w}) + 4(\alpha \cdot U)^2 \big[(\alpha \cdot \tilde{w})^2 w + (\alpha \cdot w)^2 \tilde{w} \big] \\
&\qquad + 2(\alpha \cdot U)^2 \big[|\tilde{w}|^2 w + |w|^2 \tilde{w} \big] + 4(\alpha \cdot U) \big[|w|^2 (\alpha \cdot w) \tilde{w} - |\tilde{w}|^2 (\alpha \cdot \tilde{w}) w \big] \\
&\qquad + 4(\alpha \cdot U)^3 \big[(\alpha \cdot w) \tilde{w} - (\alpha \cdot \tilde{w}) w \big] \\
&\qquad + (\alpha \cdot U)(|w|^2 + |\tilde{w}|^2) \big\{ |w|^2 - |w|^2 - 2(\alpha \cdot U) \big[\alpha \cdot (w + \tilde{w}) \big] \big\} \alpha
\end{aligned}
$$

On account of formulas (43), (44) and the lemma in Section 6 we have

$$
\begin{aligned}
&\int d_c(|w|^4 \tilde{w}) \, d\epsilon \\
&\quad = -2\pi \cos^2\theta \sin^2\theta \, |U|^4 (w + \tilde{w}) + 4\pi \cos^2\theta \sin^2\theta \, |U|^2 (|\tilde{w}|^2 w + |w|^2 \tilde{w}) \\
&\qquad - 12\pi \cos^2\theta \sin^2\theta \big[(\tilde{w} \cdot U)^2 w + (w \cdot U)^2 \tilde{w} \big] \\
&\qquad - 8\pi \cos^2\theta \sin^2\theta \, |U|^2 \big[(U \cdot w) \tilde{w} - (U \cdot \tilde{w}) w \big] \\
&\qquad - 2\pi \cos^2\theta \sin^2\theta (|w|^2 + |\tilde{w}|^2) \big\{ |U|^2 (w + \tilde{w}) - 3 \big[U \cdot (w + \tilde{w}) \big] U \big\} \\
&\quad = \pi \cos^2\theta \sin^2\theta \, \big\{ \big[4|w|^2 |\tilde{w}|^2 - 10|\tilde{w}|^4 + 2|w|^4 - 4(w \cdot \tilde{w})^2 \\
&\qquad + 4(|w|^2 + |\tilde{w}|^2)(w \cdot \tilde{w}) \big] w + \big[4|w|^2 |\tilde{w}|^2 - 10|w|^4 + 2|\tilde{w}|^4 - 4(w \cdot \tilde{w})^2 \\
&\qquad\qquad + 4(|w|^2 + |\tilde{w}|^2)(w \cdot \tilde{w}) \big] \tilde{w} \big\}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int f^{(0)} \tilde{f}^{(0)} d_c(|w|^4 \tilde{w}) B(\theta) \, d\theta \, d\epsilon \, d\tilde{v} \\
&\quad = 2B_1 n f^{(0)} \left[|w|^4 + 6 \frac{KT}{m} |w|^2 - 65 \left(\frac{KT}{m} \right)^2 \right] w
\end{aligned} \tag{94}
$$

On account of (94), (46) and the fact that w is a summational invariant, we have

$$
\begin{aligned}
&\int f^{(0)} \tilde{f}^{(0)} d_c \left\{ \left[|w|^4 - 10 \frac{KT}{m} |w|^2 + 15 \left(\frac{KT}{m} \right)^2 \right] \tilde{w} \right\} B(\theta) \, d\theta \, d\epsilon \, d\tilde{v} \\
&\quad = 2B_1 n f^{(0)} \left[|w|^4 - 14 \frac{KT}{m} |w|^2 + 35 \left(\frac{KT}{m} \right)^2 \right] w
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
&|\tilde{w}'|^4|w'|^2 + |w'|^4|\tilde{w}'|^2 - |\tilde{w}|^4|w|^2 - |w|^4|\tilde{w}|^2 \\
&\quad = |\tilde{w}'|^2|w'|^2(|\tilde{w}'|^2 + |w'|^2) - |\tilde{w}|^2|w|^2(|\tilde{w}|^2 + |w|^2) \\
&\quad = (|\tilde{w}|^2 + |w|^2)(|\tilde{w}'|^2|w'|^2 - |\tilde{w}|^2|w|^2) \\
&\quad = (|\tilde{w}|^2 + |w|^2)\{(|\tilde{w}|^2 + |w|^2)(\alpha \cdot U)^2 \\
&\qquad\qquad + 2(\alpha \cdot U)\left[|\tilde{w}|^2(\alpha \cdot w) - |w|^2(\alpha \cdot \tilde{w})\right] \\
&\qquad\qquad - (\alpha \cdot U)^4 - 4(\alpha \cdot U)^2(\alpha \cdot w)(\alpha \cdot \tilde{w})\}
\end{aligned}
$$

where we have used the equality immediately after equality (46).

On account of formulas (43), (44) and the lemma in Section 6, we have

$$
\begin{aligned}
&\int d_c(|w|^4|\tilde{w}|^2)\, d\epsilon \\
&\quad = 2\pi \cos^2\theta \sin^2\theta\left[|U|^4 - 2|U|^2(w \cdot \tilde{w}) + 6(U \cdot w)(U \cdot \tilde{w})\right](|\tilde{w}|^2 + |w|^2) \\
&\quad = 2\pi \cos^2\theta \sin^2\theta\left[|w|^4 + |\tilde{w}|^4 + 2(w \cdot \tilde{w})^2 - 4|w|^2|\tilde{w}|^2\right](|\tilde{w}|^2 + |w|^2)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int f^{(0)}\tilde{f}^{(0)} d_c(|w|^4|\tilde{w}|^2) B(\theta)\, d\theta\, d\epsilon\, d\tilde{v} \\
&\quad = 2B_1 n f^{(0)}\left[|w|^6 - \frac{7KT}{m}|w|^4 - 35\left(\frac{KT}{m}\right)^2|w|^2 + 105\left(\frac{KT}{m}\right)^3\right] \qquad (96)
\end{aligned}
$$

On account of (96), (47) and Uhlenbeck and Wang-Chang's result about the eigenvalues and eigenfunctions of the linear Boltzmann integral operator, we have

$$
\begin{aligned}
&\int f^{(0)}\tilde{f}^{(0)} d_c \left\{\left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2\right]\left(|\tilde{w}|^2 - \frac{3K\tilde{T}}{m}\right)\right\} B(\theta)\, d\theta\, d\epsilon\, d\tilde{v} \\
&\quad = 2B_1 n f^{(0)}\left[|w|^6 - 21\frac{KT}{m}|w|^4 + 105\left(\frac{KT}{m}\right)^2|w|^2 - 105\left(\frac{KT}{m}\right)^3\right] \qquad (97)
\end{aligned}
$$

Substituting the results (45), (46), (47), (86), (89), (91), (93), (95), (97) and the Uhlenbeck–Wang-Chang results about the eigenvalues and eigenfunctions of the linear Boltzmann integral operator for the corresponding terms on the right-hand side of equation (83) and arranging the sum in a proper way, we have the following expression for $J[h^{(2)}(z', \hat{z}, \tilde{z}') - h^{(2)}(z, \hat{z}, \tilde{z})]_{\tilde{x}=x}$:

$$
\begin{aligned}
&J\left[h^{(2)}(z',\hat{z},\tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})\right]_{\tilde{x}=x} \\
&\quad = B_1 n f^{(0)}\hat{f}^{(0)}\Bigg\{\sum_{\substack{1\leqslant i,s\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big(w_i w_s - \frac{1}{3}\delta_{is}|w|^2\Big) \\
&\quad\quad \times\Big[-6a_{ij}(x,\hat{x})a_{s0}(x,x) - \frac{18KT}{m}a_{ij}(x,\hat{x})a_{s4}(x,x) + 3b_{ijs}(x,\hat{x},x)\Big] \\
&\quad\quad + \sum_{\substack{1\leqslant i,s,t\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big[w_i w_s w_t - \frac{1}{5}(w_i\delta_{st} + w_s\delta_{it} + w_t\delta_{is})|w|^2\Big] \\
&\quad\quad \times\big[3a_{ij}(x,\hat{x})a_{st}(x,x)\big] + \sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big(|w|^2 - \frac{5KT}{m}\Big) \\
&\quad\quad \times w_i\Big[-\frac{2}{15}\sum_{1\leqslant k\leqslant 3} a_{kj}(x,\hat{x})a_{ki}(x,x) - \frac{2}{5}a_{ij}(x,\hat{x})\sum_{1\leqslant k\leqslant 3} a_{kk}(x,x) \\
&\quad\quad\quad - 4a_{ij}(x,\hat{x})a_{40}(x,x) - 4a_{4j}(x,\hat{x})a_{i0}(x,x) \\
&\quad\quad\quad - \frac{12KT}{m}a_{ij}(x,\hat{x})a_{44}(x,x) - \frac{12KT}{m}a_{4j}(x,\hat{x})a_{i4}(x,x) \\
&\quad\quad\quad + 4b_{ij4}(x,\hat{x},x)\Big] \\
&\quad\quad + \sum_{\substack{1\leqslant i,s\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big(|w|^4\delta_{is} - 3|w|^2 w_i w_s + \frac{21KT}{m}w_i w_s - 7|w|^2\frac{KT}{m}\delta_{is}\Big) \\
&\quad\quad \times\Big[-\frac{4}{3}a_{ij}(x,\hat{x})a_{s4}(x,x) - a_{4j}(x,\hat{x})a_{is}(x,x)\Big] \\
&\quad\quad + \sum_{0\leqslant j\leqslant 4} \hat{\psi}_j\Big[|w|^4 - 10\frac{KT}{m}|w|^2 + 15\Big(\frac{KT}{m}\Big)^2\Big] \\
&\quad\quad \times\Big[-4a_{4j}(x,\hat{x})a_{40}(x,x) - \frac{12KT}{m}a_{4j}(x,\hat{x})a_{44}(x,x) + 2b_{4j4}(x,\hat{x},x)\Big] \\
&\quad\quad + \sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big[|w|^4 - 14|w|^2\frac{KT}{m} + 35\Big(\frac{KT}{m}\Big)^2\Big] \\
&\quad\quad \times w_i\big[2a_{ij}(x,\hat{x})a_{44}(x,x) + 4a_{4j}(x,\hat{x})a_{i4}(x,x)\big] =
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{0 \leqslant j \leqslant 4} \hat{\psi}_j \left[|w|^6 - 21|w|^4 \frac{KT}{m} + 105|w|^2 \left(\frac{KT}{m} \right)^2 - 105 \left(\frac{KT}{m} \right)^3 \right] \\
&\times 2a_{4j}(x,\hat{x})a_{44}(x,x) + \sum_{1 \leqslant i,j,s,t \leqslant 3} \left(\hat{w}_i\hat{w}_s - \frac{|\hat{w}|^2}{3}\delta_{is} \right) \\
&\times \left(w_j w_t - \frac{|w|^2}{3}\delta_{jt} \right) 3a_{ij}(\hat{x},x)a_{st}(\hat{x},x) + \sum_{1 \leqslant i,j,s \leqslant 3} \left(\hat{w}_i\hat{w}_s - \frac{|\hat{w}|^2}{3}\delta_{is} \right) \\
&\times \left(|w|^2 - \frac{5KT}{m} \right) w_j 4a_{ij}(\hat{x},x)a_{s4}(\hat{x},x) + \sum_{1 \leqslant i,s \leqslant 3} \left(\hat{w}_i\hat{w}_s - \frac{|\hat{w}|^2}{3}\delta_{is} \right) \\
&\times \left[|w|^4 - 10\frac{KT}{m}|w|^2 + 15\left(\frac{KT}{m} \right)^2 \right] 2a_{i4}(\hat{x},x)a_{s4}(\hat{x},x) \\
&+ \sum_{1 \leqslant i,j,t \leqslant 3} \left(|\hat{w}|^2 - \frac{5K\hat{T}}{m} \right) \hat{w}_i \left(w_j w_t - \frac{|w|^2}{3}\delta_{jt} \right) \\
&\times \left[3a_{ij}(\hat{x},x)a_{4t}(\hat{x},x) + 3a_{4j}(\hat{x},x)a_{it}(\hat{x},x) \right] \\
&+ \sum_{1 \leqslant i,j \leqslant 3} \left(|\hat{w}|^2 - \frac{5K\hat{T}}{m} \right) \hat{w}_i \left(|w|^2 - \frac{5KT}{m} \right) \\
&\times w_j \left[4a_{ij}(\hat{x},x)a_{44}(\hat{x},x) + 4a_{4j}(\hat{x},x)a_{i4}(\hat{x},x) \right] \\
&+ \sum_{1 \leqslant i \leqslant 3} \left(|\hat{w}|^2 - \frac{5K\hat{T}}{m} \right) \hat{w}_i \left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m} \right)^2 \right] 4a_{i4}(\hat{x},x)a_{44}(\hat{x},x) \\
&+ \sum_{1 \leqslant j,t \leqslant 3} \left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m} \right)^2 \right] \left(w_j w_t - \frac{|w|^2}{3}\delta_{jt} \right) \\
&\times 3a_{4j}(\hat{x},x)a_{4t}(\hat{x},x) + \sum_{1 \leqslant j \leqslant 3} \left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m} \right)^2 \right] \\
&\times \left(|w|^2 - \frac{5KT}{m} \right) w_j 4a_{4j}(\hat{x},x)a_{44}(\hat{x},x) + \left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m} \right)^2 \right] \\
&\times \left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m} \right)^2 \right] 2\left[a_{44}(\hat{x},x) \right]^2 \Bigg\} \qquad (98)
\end{aligned}
$$

$J[h^{(2)}(z,\hat{z}',\tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})]_{\tilde{x}=\hat{x}}$ has an expression similar to that on the right-hand side of (98). More precisely, the expression for $J[h^{(2)}(z,\hat{z}', \tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})]_{\tilde{x}=\hat{x}}$ can be obtained by exchanging z and $\hat{z}$ in the expression on the right-hand side of (98).

11. THE CALCULATION OF $[f^{(1)}g^{(1)}]_{\hat{z}}$

On account of (34) and (50) we have

$$
\begin{aligned}
&g^{(1)}(z,\hat{z})f^{(1)}(\tilde{z}) \\
&= f^{(0)}\hat{f}^{(0)}\tilde{f}^{(0)} \sum_{0\leqslant j\leqslant 4} \hat{\psi}_j \sum_{0\leqslant i\leqslant 4} a_{ij}\psi_i \left\{ \sum_{1\leqslant s,t\leqslant 3} \left(\frac{\tilde{a}_{st}}{2} - \frac{m}{6B_1\tilde{n}K\tilde{T}} \frac{\partial \tilde{u}_t}{\partial \tilde{x}_s} \right) \right. \\
&\qquad \times \left(\tilde{w}_s\tilde{w}_t - \frac{1}{3}|\tilde{w}|^2\delta_{st} \right) \\
&\qquad + \sum_{1\leqslant s\leqslant 3} \left(\tilde{a}_{s4} - \frac{m}{8B_1nK\tilde{T}} \frac{\partial \tilde{T}}{\partial \tilde{x}_s} \right) \\
&\qquad \times \left(|\tilde{w}|^2\tilde{w}_s - \frac{5K\tilde{T}}{m}\tilde{w}_s \right) \\
&\qquad \left. + \frac{\tilde{a}_{44}}{2} \left[|\tilde{w}|^4 - 10\frac{K\tilde{T}}{m}|\tilde{w}|^2 + 15\left(\frac{K\tilde{T}}{m}\right)^2 \right] \right\}
\end{aligned}
\tag{99}
$$

Using the formulas (86), (89), (91), (93), (95), (97), (99), and the results about the eigenfunctions and eigenvalues of the linear Boltzmann integral operator due to Uhlenbeck and Wang-Chang, we have

$$
\begin{aligned}
\mathcal{J}[f^{(1)}g^{(1)}]_{\hat{z}} = B_1nf^{(0)}\hat{f}^{(0)} &\left\{ \sum_{\substack{1\leqslant i,s\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j \left(w_iw_s - \frac{|w|^2}{3}\delta_{is} \right) \right. \\
&\times \left[-6a_{0j}(x,\hat{x}) \left(\frac{a_{is}(x,x)}{2} - \frac{m}{6B_1nKT}\frac{\partial u_s}{\partial x_i} \right) \right. \\
&\quad \left. -18a_{4j}(x,\hat{x}) \left(\frac{a_{is}(x,x)}{2} - \frac{m}{6B_1nKT}\frac{\partial u_s}{\partial x_i} \right) \frac{KT}{m} \right] \\
&+ \sum_{\substack{1\leqslant i,s,t\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j \left[w_iw_sw_t - \frac{1}{5}(w_i\delta_{st} + w_s\delta_{it} + w_t\delta_{is})|w|^2 \right] \\
&\times \left[3a_{ij}(x,\hat{x}) \left(\frac{a_{st}(x,x)}{2} - \frac{m}{6B_1nKT}\frac{\partial u_s}{\partial x_t} \right) \right] \\
&+ \sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j \left(|w|^2 - \frac{5KT}{m} \right) =
\end{aligned}
$$

$$\times w_i\Bigg[\frac{2}{15}a_{ij}(x,\hat{x})\Bigg(\sum_{1\leqslant k\leqslant 3}a_{kk}(x,x)-\frac{m}{3B_1nKT}\sum_{1\leqslant k\leqslant 3}\frac{\partial u_k}{\partial x_k}\Bigg)$$

$$-\frac{2}{5}\sum_{1\leqslant k\leqslant 3}a_{kj}(x,\hat{x})\Bigg[a_{ik}(x,x)-\frac{m}{6B_1nKT}\Bigg(\frac{\partial u_k}{\partial x_i}+\frac{\partial u_i}{\partial x_k}\Bigg)\Bigg]$$

$$-4a_{0j}(x,\hat{x})\Bigg(a_{i4}-\frac{m}{8B_1nKT^2}\frac{\partial T}{\partial x_i}\Bigg)$$

$$-12a_{4j}(x,\hat{x})\Bigg(a_{i4}-\frac{m}{8B_1nKT^2}\frac{\partial T}{\partial x_i}\Bigg)\frac{KT}{m}\Bigg]$$

$$+\sum_{\substack{1\leqslant i,s\leqslant 3\\0\leqslant j\leqslant 4}}\hat{\psi}_j\Bigg(|w|^4\delta_{is}-3|w|^2w_iw_s+\frac{21KT}{m}w_iw_s-7|w|^2\frac{KT}{m}\delta_{is}\Bigg)$$

$$\times\Bigg[-\frac{1}{3}a_{4j}(x,\hat{x})\Bigg(\frac{a_{is}}{2}-\frac{m}{6B_1nKT}\frac{\partial u_s}{\partial x_i}\Bigg)-a_{ij}(x,\hat{x})\Bigg(a_{s4}-\frac{m}{8B_1nKT^2}\frac{\partial T}{\partial x_s}\Bigg)\Bigg]$$

$$+\sum_{0\leqslant j\leqslant 4}\hat{\psi}_j\Bigg[|w|^4-\frac{10KT}{m}|w|^2+15\Bigg(\frac{KT}{m}\Bigg)^2\Bigg]$$

$$\times\Bigg[-2a_{0j}(x,\hat{x})a_{44}(x,x)-6a_{4j}(x,\hat{x})a_{44}(x,x)\frac{KT}{m}\Bigg]$$

$$+\sum_{\substack{1\leqslant i\leqslant 3\\0\leqslant j\leqslant 4}}\hat{\psi}_j\Bigg[|w|^4-14|w|^2\frac{KT}{m}+35\Bigg(\frac{KT}{m}\Bigg)^2\Bigg]$$

$$\times w_i\Bigg[a_{ij}(x,\hat{x})a_{44}(x,x)+2a_{4j}(x,\hat{x})\Bigg(a_{i4}-\frac{m}{8B_1nKT^2}\frac{\partial T}{\partial x_i}\Bigg)\Bigg]$$

$$+\sum_{0\leqslant j\leqslant 4}\hat{\psi}_j\Bigg[|w|^6-21|w|^4\frac{KT}{m}+105|w|^2\Bigg(\frac{KT}{m}\Bigg)^2-105\Bigg(\frac{KT}{m}\Bigg)^3\Bigg]$$

$$\times a_{4j}(x,\hat{x})a_{44}(x,x)\Bigg\} \tag{100}$$

The expression for $\mathcal{J}[f^{(1)}g^{(1)}]_z$ can be obtained by exchanging z and $\hat{z}$ in the expression on the right-hand side of (100).

12. THE CALCULATION OF $\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)g^{(1)}$

The following formulas are easy to verify:

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}\right)\psi_0=0 \tag{101}$$

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}\right)\psi_i=\frac{K}{m}\frac{\partial T}{\partial x_i}+\frac{KT}{m}\frac{1}{\rho}\frac{\partial \rho}{\partial x_i}-w_k\frac{\partial u_i}{\partial x_k}\qquad (1\leqslant i\leqslant 3) \tag{102}$$

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}\right)\psi_4=2w_l\left(\frac{K}{m}\frac{\partial T}{\partial x_l}+\frac{KT}{m}\frac{1}{\rho}\frac{\partial \rho}{\partial x_l}-w_k\frac{\partial u_l}{\partial x_k}\right) \tag{103}$$

By virtue of (34), (41), (101), (102), and (103) we have

$$\begin{aligned}
&\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)g^{(1)}\\
&\quad=f^{(0)}\hat{f}^{(0)}\Bigg\{\sum_{\substack{0\leqslant i,j\leqslant 4\\ 1\leqslant k\leqslant 3}}\frac{a_{ij}}{2T}\frac{\partial T}{\partial x_k}w_k\left(\frac{m|w|^2}{KT}-5\right)\psi_i\hat{\psi}_j\\
&\qquad+\sum_{\substack{0\leqslant i,j\leqslant 4\\ 1\leqslant k,l\leqslant 3}}\frac{m}{KT}\frac{\partial u_k}{\partial x_l}a_{ij}\left(w_kw_l-\frac{1}{3}|w|^2\delta_{kl}\right)\psi_i\hat{\psi}_j\\
&\qquad+\sum_{\substack{0\leqslant i,j\leqslant 4\\ 1\leqslant k\leqslant 3}}\frac{a_{ij}}{2\hat{T}}\frac{\partial \hat{T}}{\partial \hat{x}_k}\hat{w}_k\left(\frac{m|\hat{w}|^2}{K\hat{T}}-5\right)\psi_i\hat{\psi}_j\\
&\qquad+\sum_{\substack{0\leqslant i,j\leqslant 4\\ 1\leqslant k,l\leqslant 3}}\frac{m}{K\hat{T}}\frac{\partial \hat{u}_k}{\partial \hat{x}_l}a_{ij}\left(\hat{w}_k\hat{w}_l-\frac{1}{3}|\hat{w}|^2\delta_{kl}\right)\psi_i\hat{\psi}_j\\
&\qquad+\sum_{0\leqslant i,j\leqslant 4}\left[\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{ij}\right]\psi_i\hat{\psi}_j\\
&\qquad+\sum_{\substack{1\leqslant j\leqslant 3\\ 0\leqslant i\leqslant 4}}a_{ij}\psi_i\left(\frac{K}{m}\frac{\partial \hat{T}}{\partial \hat{x}_j}+\frac{K\hat{T}}{m}\frac{1}{\hat{\rho}}\frac{\partial \hat{\rho}}{\partial \hat{x}_j}-\hat{w}_k\frac{\partial \hat{u}_j}{\partial \hat{x}_k}\right)\\
&\qquad+2\sum_{0\leqslant i\leqslant 4}a_{i4}\psi_i\left(\frac{K}{m}\frac{\partial \hat{T}}{\partial \hat{x}_k}+\frac{K\hat{T}}{m}\frac{1}{\hat{\rho}}\frac{\partial \hat{\rho}}{\partial \hat{x}_k}-\hat{w}_l\frac{\partial \hat{u}_k}{\partial \hat{x}_l}\right)\hat{w}_k=
\end{aligned}$$

$$+ \sum_{\substack{1 \leqslant i \leqslant 3 \\ 0 \leqslant j \leqslant 4}} a_{ij} \hat{\psi}_j \left(\frac{K}{m} \frac{\partial T}{\partial x_i} + \frac{KT}{m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} - w_k \frac{\partial u_i}{\partial x_k} \right)$$

$$+ 2 \sum_{0 \leqslant j \leqslant 4} a_{4j} \hat{\psi}_j \left(\frac{K}{m} \frac{\partial T}{\partial x_k} + \frac{KT}{m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_k} - w_l \frac{\partial u_k}{\partial x_l} \right) w_k \Bigg\} \quad (104)$$

In order to get an explicit expression for

$$\left(\frac{\partial_0}{\partial t} + v_k \frac{\partial}{\partial x_k} + \hat{v}_k \frac{\partial}{\partial \hat{x}_k} \right) g^{(1)}$$

we have to calculate

$$\left(\frac{\partial_0}{\partial t} + v_k \frac{\partial}{\partial x_k} + \hat{v}_k \frac{\partial}{\partial \hat{x}_k} \right) a_{ij} \qquad (0 \leqslant i, j \leqslant 4)$$

A simple calculation shows us that

$$\left(\frac{\partial_0}{\partial t} + v_k \frac{\partial}{\partial x_k} + \hat{v}_k \frac{\partial}{\partial \hat{x}_k} \right) a_{ij}$$

$$= \left(w_k \frac{\partial}{\partial x_k} + \hat{w}_k \frac{\partial}{\partial \hat{x}_k} \right) a_{ij} + \frac{2}{3} a_{ij} \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) - a_{kj} \frac{\partial u_i}{\partial x_k} - a_{ik} \frac{\partial \hat{u}_j}{\partial \hat{x}_k}$$

$$- \frac{\partial a_{i0}}{\partial \hat{x}_j} - \frac{\partial a_{0j}}{\partial x_i} - \frac{5K\hat{T}}{m} \frac{\partial a_{i4}}{\partial \hat{x}_j} - \frac{5KT}{m} \frac{\partial a_{4j}}{\partial x_i} - \frac{7K}{m} a_{i4} \frac{\partial \hat{T}}{\partial \hat{x}_j} - \frac{7K}{m} a_{4j} \frac{\partial T}{\partial x_i}$$

$$- \frac{2K\hat{T}}{m} a_{i4} \frac{1}{\hat{\rho}} \frac{\partial \hat{\rho}}{\partial \hat{x}_j} - \frac{2KT}{m} a_{4j} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} \qquad (1 \leqslant i, j \leqslant 3) \qquad (105)$$

$$\left(\frac{\partial_0}{\partial t} + v_k \frac{\partial}{\partial x_k} + \hat{v}_k \frac{\partial}{\partial \hat{x}_k} \right) a_{i0}$$

$$= \left(w_k \frac{\partial}{\partial x_k} + \hat{w}_k \frac{\partial}{\partial \hat{x}_k} \right) a_{i0} + \frac{2}{3} a_{i0} \frac{\partial u_k}{\partial x_k} - a_{k0} \frac{\partial u_i}{\partial x_k}$$

$$- 2a_{40} \frac{KT}{m} \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} + a_{ij} \left(\frac{3}{2} \frac{K}{m} \frac{\partial \hat{T}}{\partial \hat{x}_j} - \frac{K\hat{T}}{m} \frac{1}{\hat{\rho}} \frac{\partial \hat{\rho}}{\partial \hat{x}_j} \right)$$

$$- \left(\frac{\partial a_{00}}{\partial x_i} + \frac{5KT}{m} \frac{\partial a_{40}}{\partial x_i} + \frac{7K}{m} a_{40} \frac{\partial T}{\partial x_i} \right) \qquad (1 \leqslant i \leqslant 3)$$

(106)

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{i4}$$

$$=\left(w_k\frac{\partial}{\partial x_k}+\hat{w}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{i4}+\frac{2}{3}a_{i4}\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)-a_{k4}\frac{\partial u_i}{\partial x_k}$$

$$-\left(\frac{\partial a_{04}}{\partial x_i}+\frac{5KT}{m}\frac{\partial a_{44}}{\partial x_i}+\frac{7K}{m}a_{44}\frac{\partial T}{\partial x_i}\right)-\frac{1}{3}\frac{\partial a_{ik}}{\partial \hat{x}_k}-\frac{5}{6}a_{ik}\frac{1}{\hat{T}}\frac{\partial \hat{T}}{\partial \hat{x}_k}$$

$$-\frac{2KT}{m}a_{44}\frac{1}{\rho}\frac{\partial \rho}{\partial x_i}\qquad (1\leqslant i\leqslant 3)\tag{107}$$

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{00}$$

$$=\left(w_k\frac{\partial}{\partial x_k}+\hat{w}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{00}+\frac{3K}{2m}\left(\frac{\partial T}{\partial x_k}a_{k0}+\frac{\partial \hat{T}}{\partial \hat{x}_k}a_{0k}\right)$$

$$-\frac{K}{m}\left(\frac{T}{\rho}\frac{\partial \rho}{\partial x_k}a_{k0}+\frac{\hat{T}}{\hat{\rho}}\frac{\partial \hat{\rho}}{\partial \hat{x}_k}a_{0k}\right)\tag{108}$$

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{04}$$

$$=\left(w_k\frac{\partial}{\partial x_k}+\hat{w}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{04}+\frac{2}{3}a_{04}\frac{\partial \hat{u}_k}{\partial \hat{x}_k}$$

$$+\frac{a_{k4}}{2}\left(\frac{3K}{m}\frac{\partial T}{\partial x_k}-\frac{2KT}{m}\frac{1}{\rho}\frac{\partial \rho}{\partial x_k}\right)-\frac{5}{6}a_{0k}\frac{1}{\hat{T}}\frac{\partial \hat{T}}{\partial \hat{x}_k}-\frac{1}{3}\frac{\partial a_{0k}}{\partial \hat{x}_k}\tag{109}$$

$$\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{44}$$

$$=\left(w_k\frac{\partial}{\partial x_k}+\hat{w}_k\frac{\partial}{\partial \hat{x}_k}\right)a_{44}+\frac{2}{3}a_{44}\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)-\frac{5}{6}a_{k4}\frac{1}{T}\frac{\partial T}{\partial x_k}$$

$$-\frac{5}{6}a_{4k}\frac{1}{\hat{T}}\frac{\partial \hat{T}}{\partial \hat{x}_k}-\frac{1}{3}\frac{\partial a_{k4}}{\partial x_k}-\frac{1}{3}\frac{\partial a_{4k}}{\partial \hat{x}_k}\tag{110}$$

Substituting the right-hand sides of the equalities (105), (106), (107), (108), (109), and (110) for the corresponding terms on the right-hand side of (104), we can get the following formula through an elementary, but slightly clumsy, calculation:

$$
\begin{aligned}
&\left(\frac{\partial_0}{\partial t}+v_k\frac{\partial}{\partial x_k}+\hat{v}_k\frac{\partial}{\partial \hat{x}_k}\right)g^{(1)}\\
&=f^{(0)}\hat{f}^{(0)}\Bigg\{\sum_{\substack{1\leqslant i,k\leqslant 3\\ 0\leqslant j\leqslant 4}}\hat{\psi}_j\left(ww_k-\frac{\delta_{ik}}{3}|w|^2\right)\\
&\quad\times\left[\frac{a_{ij}(x,\hat{x})}{T}\frac{\partial T}{\partial x_k}+\frac{m}{KT}a_{0j}(x,\hat{x})\frac{\partial u_i}{\partial x_k}+5a_{4j}(x,\hat{x})\frac{\partial u_i}{\partial x_k}+\frac{\partial a_{ij}(x,\hat{x})}{\partial x_k}\right]\\
&\quad+\sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 4}}\hat{\psi}_j\left(|w|^2-\frac{5KT}{m}\right)w_i\Bigg[\frac{m}{2KT^2}a_{0j}(x,\hat{x})\frac{\partial T}{\partial x_i}+\frac{9a_{4j}(x,\hat{x})}{2T}\frac{\partial T}{\partial x_i}\\
&\qquad-\frac{2m}{15KT}a_{ij}(x,\hat{x})\sum_{1\leqslant k\leqslant 3}\frac{\partial u_k}{\partial x_k}\\
&\qquad+\frac{m}{5KT}\sum_{1\leqslant k\leqslant 3}a_{kj}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)+\frac{\partial a_{4j}(x,\hat{x})}{\partial x_i}\Bigg]\\
&\quad+\sum_{\substack{1\leqslant i,k,l\leqslant 3\\ 0\leqslant j\leqslant 4}}\hat{\psi}_j\left[w_iw_kw_l-\frac{1}{5}(w_i\delta_{kl}+w_k\delta_{il}+w_l\delta_{ik})|w|^2\right]\\
&\quad\times a_{ij}(x,\hat{x})\frac{\partial u_k}{\partial x_l}\frac{m}{KT}+\sum_{0\leqslant j\leqslant 4}\hat{\psi}_j\left[|w|^4-\frac{10KT}{m}|w|^2+15\left(\frac{KT}{m}\right)^2\right]\\
&\quad\times\frac{m}{6KT^2}\sum_{1\leqslant k\leqslant 3}a_{kj}(x,\hat{x})\frac{\partial T}{\partial x_k}\\
&\quad+\sum_{\substack{1\leqslant k,l\leqslant 3\\ 0\leqslant j\leqslant 4}}\hat{\psi}_j\left(|w|^4\delta_{kl}-3|w|^2w_kw_l-\frac{7KT}{m}|w|^2\delta_{kl}+\frac{21KT}{m}w_kw_l\right)\\
&\quad\times\left[-\frac{m}{6KT^2}a_{kj}(x,\hat{x})\frac{\partial T}{\partial x_l}-\frac{m}{3KT}a_{4j}(x,\hat{x})\frac{\partial u_k}{\partial x_l}\right]\\
&\quad+\sum_{\substack{1\leqslant k\leqslant 3\\ 0\leqslant j\leqslant 4}}\hat{\psi}_j\left[|w|^4-14|w|^2\frac{KT}{m}+35\left(\frac{KT}{m}\right)^2\right]\\
&\quad\times w_k\frac{m}{2KT^2}a_{4j}(x,\hat{x})\frac{\partial T}{\partial x_k}\Bigg\}
\end{aligned}
$$

+ the expression obtained from the above expression by exchanging z and $\hat{z}$ (111)

13. THE SOLUTION OF EQUATION (31)

Substituting the right-hand sides of (98), (100), and (111) for the corresponding terms on the right-hand side of the equation (31), we have

$$
\begin{aligned}
&\mathcal{J}[f^{(0)}g^{(2)}]_z + \mathcal{J}[f^{(0)}g^{(0)}]_{\hat z} \\
&= B_1 n f^{(0)}\hat f^{(0)}\Bigg\{ \sum_{\substack{1\le i,k\le 3\\ 0\le j\le 4}} \hat\psi_j\left(w_i w_k - \frac{\delta_{ik}}{3}|w|^2\right) \\
&\times \Bigg[6a_{ij}(x,\hat x)a_{k0}(x,x) + \frac{18KT}{m} a_{ij}(x,\hat x)a_{k4}(x,x) \\
&\quad + 3a_{0j}(x,\hat x)a_{ik}(x,x) + \frac{9KT}{m} a_{4j}(x,\hat x)a_{ik}(x,x) - 3b_{ijk}(x,\hat x,x) \\
&\quad + \frac{a_{4j}(x,\hat x)}{B_1 n}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \frac{1}{B_1 n T}\frac{\partial[T a_{ij}(x,\hat x)]}{\partial x_k}\Bigg] \\
&+ \sum_{\substack{1\le i\le 3\\ 0\le j\le 4}} \hat\psi_j\left(|w|^2 - \frac{5KT}{m}\right) w_i \Bigg[\frac{8}{15}\sum_{1\le k\le 3} a_{kj}(x,\hat x)a_{ki}(x,x) \\
&+ \frac{4}{15} a_{ij}(x,\hat x)\sum_{1\le k\le 3} a_{kk}(x,x) + 4a_{ij}(x,\hat x)a_{40}(x,x) \\
&+ 4a_{4j}(x,\hat x)a_{i0}(x,x) + \frac{12KT}{m} a_{ij}(x,\hat x)a_{44}(x,x) \\
&+ \frac{24KT}{m} a_{4j}(x,\hat x)a_{i4}(x,x) + 4a_{0j}(x,\hat x)a_{i4}(x,x) \\
&- 4b_{ij4}(x,\hat x,x) - \frac{4m}{45B_1 nKT} a_{ij}(x,\hat x)\sum_{1\le k\le 3}\frac{\partial u_k}{\partial x_k} \\
&+ \frac{2m}{15B_1 nKT}\sum_{1\le k\le 3} a_{kj}(x,\hat x)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) + \frac{3}{B_1 nT} a_{4j}(x,\hat x)\frac{\partial T}{\partial x_i} \\
&+ \frac{1}{B_1 n}\frac{\partial a_{4j}(x,\hat x)}{\partial x_i}\Bigg] =
\end{aligned}
$$

$$+\sum_{\substack{1\leqslant i,k,l\leqslant 3\\0\leqslant j\leqslant 4}}\hat{\psi}_j\left[w_iw_kw_l-\frac{1}{5}(w_i\delta_{kl}+w_k\delta_{il}+w_l\delta_{ik})|w|^2\right]$$

$$\times\left[-\frac{9}{2}a_{ij}(x,\hat{x})a_{kl}(x,x)+\frac{3m}{2B_1nKT}a_{ij}(x,\hat{x})\frac{\partial u_k}{\partial x_l}\right]$$

$$+\sum_{0\leqslant j\leqslant 4}\hat{\psi}_j\left[|w|^4-\frac{10KT}{m}|w|^2+15\left(\frac{KT}{m}\right)^2\right]$$

$$\times\left[4a_{ij}(x,\hat{x})a_{40}(x,x)+\frac{18KT}{m}a_{4j}(x,\hat{x})a_{44}(x,x)+2a_{0j}(x,\hat{x})a_{44}(x,x)\right.$$

$$\left.-2b_{4j4}(x,\hat{x},x)+\frac{m}{6B_1nKT^2}\sum_{1\leqslant k\leqslant 3}a_{kj}(x,\hat{x})\frac{\partial T}{\partial x_k}\right]$$

$$+\sum_{\substack{1\leqslant k,l\leqslant 3\\0\leqslant j\leqslant 4}}\hat{\psi}_j\left[|w|^4\delta_{kl}-3|w|^2w_kw_l-\frac{7KT}{m}|w|^2\delta_{kl}+\frac{21KT}{m}w_kw_l\right]$$

$$\times\left[\frac{7}{3}a_{kj}(x,\hat{x})a_{l4}(x,x)+\frac{7}{6}a_{4j}(x,\hat{x})a_{kl}(x,x)\right.$$

$$\left.-\frac{7m}{36B_1nKT}a_{4j}(x,\hat{x})\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)-\frac{7m}{24B_1nKT^2}a_{kj}(x,\hat{x})\frac{\partial T}{\partial x_l}\right]$$

$$+\sum_{\substack{1\leqslant k\leqslant 3\\0\leqslant j\leqslant 4}}\hat{\psi}_j\left[|w|^4-\frac{14KT}{m}|w|^2+35\left(\frac{KT}{m}\right)^2\right]$$

$$\times w_k\left[-3a_{kj}(x,\hat{x})a_{44}(x,x)-6a_{4j}(x,\hat{x})a_{k4}(x,x)+\frac{3m}{4B_1nKT^2}a_{4j}(x,\hat{x})\frac{\partial T}{\partial x_k}\right]$$

$$+\sum_{0\leqslant j\leqslant 4}\hat{\psi}_j\left[|w|^6-\frac{21KT}{m}|w|^4+105\left(\frac{KT}{m}\right)^2|w|^2-105\left(\frac{KT}{m}\right)^3\right]$$

$$\left[-3a_{4j}(x,\hat{x})a_{44}(x,x)\right]$$

$$+\sum_{1\leqslant i,j,s,t\leqslant 3}\left(w_jw_t-\frac{1}{3}|w|^2\delta_{jt}\right)\left(\hat{w}_i\hat{w}_s-\frac{1}{3}|\hat{w}|^2\delta_{is}\right)\left[-3a_{ij}(\hat{x},x)a_{st}(\hat{x},x)\right]=$$

$$+\sum_{1\leqslant i,j,s\leqslant 3}\left(|w|^2-\frac{5KT}{m}\right)w_j\left(\hat{w}_i\hat{w}_s-\frac{1}{3}|\hat{w}|^2\delta_{is}\right)\left[-4a_{ij}(\hat{x},x)a_{s4}(\hat{x},x)\right]$$

$$+\sum_{1\leqslant i,s\leqslant 3}\left[|w|^4-\frac{10KT}{m}|w|^2+15\left(\frac{KT}{m}\right)^2\right]$$

$$\times\left(\hat{w}_i\hat{w}_s-\frac{1}{3}|\hat{w}|^2\delta_{is}\right)\left[-2a_{i4}(\hat{x},x)a_{s4}(\hat{x},x)\right]$$

$$+\sum_{1\leqslant i,j,t\leqslant 3}\left(w_jw_t-\frac{|w|^2}{3}\delta_{jt}\right)\left(|\hat{w}|^2-\frac{5K\hat{T}}{m}\right)\hat{w}_i\left[-6a_{ij}(\hat{x},x)a_{4t}(\hat{x},x)\right]$$

$$+\sum_{1\leqslant i,j\leqslant 3}\left(|w|^2-\frac{5KT}{m}\right)w_j\left(|\hat{w}|^2-\frac{5K\hat{T}}{m}\right)$$

$$\times\hat{w}_i\left[-4a_{ij}(\hat{x},x)a_{44}(\hat{x},x)-4a_{4j}(\hat{x},x)a_{i4}(\hat{x},x)\right]$$

$$+\sum_{1\leqslant i\leqslant 3}\left[|w|^4-\frac{10KT}{m}|w|^2+15\left(\frac{KT}{m}\right)^2\right]\left(|\hat{w}|^2-\frac{5K\hat{T}}{m}\right)$$

$$\times\hat{w}_i\left[-4a_{i4}(\hat{x},x)a_{44}(\hat{x},x)\right]+\sum_{1\leqslant j,t\leqslant 3}\left(w_jw_t-\frac{1}{3}|w|^2\delta_{jt}\right)$$

$$\times\left[|\hat{w}|^4-\frac{10K\hat{T}}{m}|\hat{w}|^2+15\left(\frac{K\hat{T}}{m}\right)^2\right]\left[-3a_{4j}(\hat{x},x)a_{4t}(\hat{x},x)\right]$$

$$+\sum_{1\leqslant j\leqslant 3}\left(|w|^2-\frac{5KT}{m}\right)w_j\left[|\hat{w}|^4-\frac{10K\hat{T}}{m}|\hat{w}|^2+15\left(\frac{K\hat{T}}{m}\right)^2\right]$$

$$\times\left[-4a_{4j}(\hat{x},x)a_{44}(\hat{x},x)\right]+\left[|w|^4-\frac{10KT}{m}|w|^2+15\left(\frac{KT}{m}\right)^2\right]$$

$$\times\left[|\hat{w}|^4-\frac{10K\hat{T}}{m}|\hat{w}|^2+15\left(\frac{K\hat{T}}{m}\right)^2\right]\left\{-2\left[a_{44}(\hat{x},x)\right]^2\right\}\Bigg\}$$

+ the expression obtained from the above one by exchanging z and $\hat{z}$

(112)

Wang-Chang and Uhlenbeck[2] have found all the eigenfunctions with their corresponding eigenvalues for the linear Boltzmann integral operator.

Some of them, which we shall use in the sequel, are listed as follows:

Eigenfunctions	*The corresponding eigenvalues*
$f^{(0)}$	0
$f^{(0)} w_i \qquad (1 \leqslant i \leqslant 3)$	0
$f^{(0)} \lvert w\rvert^2$	0
$f^{(0)}\left(w_i w_j - \frac{1}{3}\lvert w\rvert^2 \delta_{ij}\right) \qquad (1 \leqslant i, j \leqslant 3)$	$-6B_1 n$
$f^{(0)}\left(\lvert w\rvert^2 - \frac{5KT}{m}\right) w_i \qquad (1 \leqslant i \leqslant 3)$	$-4B_1 n$
$f^{(0)}\left[w_i w_j w_k - \frac{1}{5}\left(w_i \delta_{jk} + w_j \delta_{ik} + w_k \delta_{ij}\right)\lvert w\rvert^2\right]$ $(1 \leqslant i, j, k \leqslant 3)$	$-9B_1 n$
$f^{(0)}\left[\lvert w\rvert^4 - 10\frac{KT}{m}\lvert w\rvert^2 + 15\left(\frac{KT}{m}\right)^2\right]$	$-4B_1 n$
$f^{(0)}\left(\lvert w\rvert^4 \delta_{ij} - 3\lvert w\rvert^2 w_i w_j - 7\frac{KT}{m}\lvert w\rvert^2 \delta_{ij} + 21\frac{KT}{m} w_i w_j\right)$ $(1 \leqslant i, j \leqslant 3)$	$-7B_1 n$
$f^{(0)}\left[\lvert w\rvert^4 - 14\frac{KT}{m}\lvert w\rvert^2 + 35\left(\frac{KT}{m}\right)^2\right] w_i$ $(1 \leqslant i \leqslant 3)$	$-6B_1 n$
$f^{(0)}\left[\lvert w\rvert^6 - 21\frac{KT}{m}\lvert w\rvert^4 + 105\left(\frac{KT}{m}\right)^2\lvert w\rvert^2 - 105\left(\frac{KT}{m}\right)^3\right]$	$-6B_1 n$

Each tensor product of two eigenfunctions of the linear Boltzmann integral operator is an eigenfunction of the linear Boltzmann double-integral operator with the sum of two corresponding eigenvalues as its own eigenvalue. For instance,

$$\mathcal{J}\left[f^{(0)}\hat{f}^{(0)}\tilde{f}^{(0)}\left(w_i w_j - \frac{1}{3}\lvert w\rvert^2 \delta_{ij}\right)\left(\lvert \hat{w}\rvert^2 - \frac{5K\hat{T}}{m}\right)\hat{w}_k\right]_z$$

$$+\mathcal{J}\left[f^{(0)}\hat{f}^{(0)}\tilde{f}^{(0)}\left(w_i w_j - \frac{1}{3}\lvert w\rvert^2 \delta_{ij}\right)\left(\lvert \hat{w}\rvert^2 - \frac{5K\hat{T}}{m}\right)w_k\right]_{\hat{z}}$$

$$= (-6B_1 n - 4B_1 \hat{n}) f^{(0)}\hat{f}^{(0)}\left(w_i w_j - \frac{1}{3}\lvert w\rvert^2 \delta_{ij}\right)\left(\lvert \hat{w}\rvert^2 - \frac{5K\hat{T}}{m}\right)\hat{w}_k$$

Using the above facts, we find a solution of equation (112) as follows:

$$g^{(2)} = f^{(0)}\hat{f}^{(0)}\Bigg\{ \sum_{\substack{1\leqslant i,k\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big(w_i w_k - \frac{1}{3}\delta_{ik}|w|^2\Big)$$

$$\times\Bigg[-a_{ij}(x,\hat{x})a_{k0}(x,x) - \frac{3KT}{m}a_{ij}(x,\hat{x})a_{k4}(x,x) - \frac{1}{2}a_{0j}(x,\hat{x})a_{ik}(x,x)$$

$$- \frac{3KT}{2m}a_{4j}(x,\hat{x})a_{ik}(x,x) + \frac{1}{2}b_{ijk}(x,\hat{x},x)$$

$$- \frac{a_{4j}(x,\hat{x})}{6B_1 n}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{6B_1 nT}\frac{\partial[Ta_{ij}(x,\hat{x})]}{\partial x_k}\Bigg]$$

$$+ \sum_{\substack{1\leqslant i\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big(|w|^2 - \frac{5KT}{m}\Big)$$

$$\times w_i\Bigg[-\frac{2}{15}\sum_{1\leqslant k\leqslant 3} a_{kj}(x,\hat{x})a_{ki}(x,x) - \frac{1}{15}a_{ij}(x,\hat{x})\sum_{1\leqslant k\leqslant 3} a_{kk}(x,x)$$

$$- a_{ij}(x,\hat{x})a_{40}(x,x) - a_{4j}(x,\hat{x})a_{i0}(x,x) - \frac{3KT}{m}a_{ij}(x,\hat{x})a_{44}(x,x)$$

$$- \frac{6KT}{m}a_{4j}(x,\hat{x})a_{i4}(x,x) + b_{ij4}(x,\hat{x},x) + \frac{m}{45B_1 nKT}a_{ij}(x,\hat{x})$$

$$\times \sum_{1\leqslant k\leqslant 3}\frac{\partial u_k}{\partial x_k} - \frac{m}{30B_1 nKT}\sum_{1\leqslant k\leqslant 3} a_{kj}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$- a_{0j}(x,\hat{x})a_{i4}(x,x) - \frac{3}{4B_1 nT}a_{4j}(x,\hat{x})\frac{\partial T}{\partial x_i} - \frac{1}{4B_1 n}\frac{\partial a_{4j}(x,\hat{x})}{\partial x_i}\Bigg]$$

$$+ \sum_{\substack{1\leqslant i,k,l\leqslant 3\\ 0\leqslant j\leqslant 4}} \hat{\psi}_j\Big[w_i w_k w_l - \frac{1}{5}(w_i\delta_{kl} + w_k\delta_{il} + w_l\delta_{ik})|w|^2\Big]$$

$$\times\left[\frac{1}{2}a_{ij}(x,\hat{x})a_{kl}(x,x) - \frac{m}{6B_1 nKT}a_{ij}(x,\hat{x})\frac{\partial u_x}{\partial x_l}\right] =$$

$$+ \sum_{0 \le j \le 4} \hat{\psi}_j \left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2 \right]$$

$$\times \left[-a_{4j}(x,\hat{x})a_{40}(x,x) - \frac{9KT}{2m} a_{4j}(x,\hat{x})a_{44}(x,x) - \frac{1}{2} a_{0j}(x,\hat{x})a_{44}(x,x) \right.$$

$$\left. + \frac{1}{2} b_{4j4}(x,\hat{x},x) - \frac{m}{24B_1 nKT^2} \sum_{1 \le k \le 3} a_{kj}(x,\hat{x}) \frac{\partial T}{\partial x_k} \right]$$

$$+ \sum_{\substack{1 \le k,l \le 3 \\ 0 \le j \le 4}} \hat{\psi}_j \left[|w|^4 \delta_{kl} - 3|w|^2 w_k w_l - \frac{7KT}{m}|w|^2 \delta_{kl} + \frac{21KT}{m} w_k w_l \right]$$

$$\times \left[-\frac{1}{3} a_{kj}(x,\hat{x})a_{l4}(x,x) - \frac{1}{6} a_{4j}(x,\hat{x})a_{kl}(x,x) \right.$$

$$\left. + \frac{m}{36B_1 nKT} a_{4j}(x,\hat{x}) \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) + \frac{m}{24B_1 nKT^2} a_{kj}(x,x) \frac{\partial T}{\partial x_l} \right]$$

$$+ \sum_{\substack{1 \le k \le 3 \\ 0 \le j \le 4}} \hat{\psi}_j \left[|w|^4 - \frac{14KT}{m}|w|^2 + 35\left(\frac{KT}{m}\right)^2 \right]$$

$$\times w_k \left[\frac{1}{2} a_{kj}(x,\hat{x})a_{44}(x,x) + a_{4j}(x,\hat{x})a_{k4}(x,x) - \frac{m}{8B_1 nKT^2} a_{4j}(x,\hat{x}) \frac{\partial T}{\partial x_k} \right]$$

$$+ \sum_{0 \le j \le 4} \hat{\psi}_j \left[|w|^6 - 21\frac{KT}{m}|w|^4 + 105\left(\frac{KT}{m}\right)^2 |w|^2 - 105\left(\frac{KT}{m}\right)^3 \right]$$

$$\times \left[\frac{1}{2} a_{4j}(x,\hat{x})a_{44}(x,x) \right]$$

+ the expression obtained from the above expression by exchanging z and $\hat{z}$

$$+ \sum_{1 \le i,j,s,t \le 3} \left(w_j w_t - \frac{1}{3}|w|^2 \delta_{jt} \right) \left(\hat{w}_i \hat{w}_s - \frac{1}{3}|\hat{w}|^2 \delta_{is} \right) \frac{1}{2} a_{ij}(\hat{x},x) a_{st}(\hat{x},x)$$

$$+ \sum_{1 \le i,j,s \le 3} \left(|w|^2 - \frac{5KT}{m} \right) w_j \left(\hat{w}_i \hat{w}_s - \frac{|\hat{w}|^2}{3} \delta_{is} \right) a_{ij}(\hat{x},x) a_{s4}(\hat{x},x)$$

$$+ \sum_{1 \le i,s \le 3} \left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2 \right] \left(\hat{w}_i \hat{w}_s - \frac{1}{3}|\hat{w}|^2 \delta_{is} \right)$$

$$\times \frac{1}{2} a_{i4}(\hat{x},x) a_{s4}(\hat{x},x) + \sum_{1 \le i,j,t \le 3} \left(w_j w_t - \frac{|w|^2}{3} \delta_{jt} \right) \left(|\hat{w}|^2 - \frac{5K\hat{T}}{m} \right) =$$

$$\times \hat{w}_i a_{ij}(\hat{x},x)a_{4t}(\hat{x},x) + \sum_{1\leqslant i,j\leqslant 3}\left(|w|^2 - \frac{5KT}{m}\right)w_i\left(|\hat{w}|^2 - \frac{5K\hat{T}}{m}\right)$$

$$\times \hat{w}_i[a_{ij}(\hat{x},x)a_{44}(\hat{x},x) + a_{4j}(\hat{x},x)a_{i4}(\hat{x},x)]$$

$$+ \sum_{1\leqslant i\leqslant 3}\left[|w|^4 - 10\frac{KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2\right]\left(|\hat{w}|^2 - \frac{5K\hat{T}}{m}\right)$$

$$\times \hat{w}_i[a_{i4}(\hat{x},x)a_{44}(\hat{x},x)] + \sum_{1\leqslant j,t\leqslant 3}\left(w_j w_t - \frac{|w|^2}{3}\delta_{jt}\right)$$

$$\times\left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m}\right)^2\right]\frac{1}{2}a_{4j}(\hat{x},x)a_{4t}(\hat{x},x)$$

$$+ \sum_{1\leqslant j\leqslant 3}\left(|w|^2 - \frac{5KT}{m}\right)w_j\left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m}\right)^2\right]$$

$$\times a_{4j}(\hat{x},x)a_{44}(\hat{x},x) + \left[|w|^4 - \frac{10KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2\right]$$

$$\times\left[|\hat{w}|^4 - \frac{10K\hat{T}}{m}|\hat{w}|^2 + 15\left(\frac{K\hat{T}}{m}\right)^2\right]\frac{1}{2}[a_{44}(\hat{x},x)]^2\Bigg\} \tag{113}$$

Obviously the above solution satisfies the additional integral conditions.

14. APPROXIMATE EQUATIONS OF ORDER $1\frac{2}{3}$ FOR TURBULENT FLOWS

The summation convention will be used throughout this section. On account of (72) we have the following equalities:

$$S_{ij}^{(2,0,0)(0)} = \frac{1}{n}\left[R_i^{(1,0)}(x,\hat{x})R_j^{(1,0)}(x,\check{x}) + R_j^{(1,0)}(x,\hat{x})R_i^{(1,0)}(x,\check{x})\right]$$

$$+ \frac{\delta_{ij}}{3}\left[S^{(2,0,0)} - \frac{1}{n}R_k^{(1,0)}(x,\hat{x})R_k^{(1,0)}(x,\check{x})\right]$$

$$S_{ij,k}^{(2,1,0)(0)} = \frac{1}{n}\left[R_{i,k}^{(1,1)}(x,\hat{x})R_j^{(1,0)}(x,\check{x}) + R_{j,k}^{(1,1)}(x,\hat{x})R_i^{(1,0)}(x,\check{x})\right]$$

$$+ \frac{\delta_{ij}}{3}\left[S_k^{(2,1,0)} - \frac{2}{n}R_{l,k}^{(1,1)}(x,\hat{x})R_l^{(1,0)}(x,\check{x})\right] =$$

$$S^{(2,1,1)(0)}_{ij,k,l} = \frac{1}{n}\left[R^{(1,1)}_{i,k}(x,\hat{x})R^{(1,1)}_{j,l}(x,\check{x}) + R^{(1,1)}_{j,k}(x,\hat{x})R^{(1,1)}_{i,l}(x,\check{x})\right] + \frac{\delta_{ij}}{3}\left[S^{(2,1,1)}_{k,l} - \frac{2}{n}R^{(1,1)}_{m,k}(x,\hat{x})R^{(1,1)}_{m,l}(x,\check{x})\right]$$

$$S^{(2,2,0)(0)}_{ij} = \frac{1}{n}\left[R^{(1,2)}_{i}(x,\hat{x})R^{(1,0)}_{j}(x,\check{x}) + R^{(1,2)}_{j}(x,\hat{x})R^{(1,0)}_{i}(x,\check{x})\right] + \frac{\delta_{ij}}{3}\left[S^{(2,2,0)} - \frac{2}{n}R^{(1,2)}_{k}(x,\hat{x})R^{(1,0)}_{k}(x,\check{x})\right]$$

$$S^{(2,2,1)(0)}_{ij,\ k} = \frac{1}{n}\left[R^{(1,2)}_{i}(x,\hat{x})R^{(1,1)}_{j,k}(x,\check{x}) + R^{(1,2)}_{j}(x,\hat{x})R^{(1,1)}_{i,k}(x,\check{x})\right] + \frac{\delta_{ij}}{3}\left[S^{(2,2,1)}_{k} - \frac{2}{n}R^{(1,2)}_{l}(x,\hat{x})R^{(1,1)}_{l,k}(x,\check{x})\right]$$

$$S^{(2,2,2)(0)}_{ij} = \frac{1}{n}\left[R^{(1,2)}_{i}(x,\hat{x})R^{(1,2)}_{j}(x,\check{x}) + R^{(1,2)}_{j}(x,\hat{x})R^{(1,2)}_{i}(x,\check{x})\right] + \frac{\delta_{ij}}{3}\left[S^{(2,2,2)} - \frac{2}{n}R^{(1,2)}_{k}(x,\hat{x})R^{(1,2)}_{k}(x,\check{x})\right]$$

$$S^{(3,0,0)(0)}_{i} = \frac{5KT}{m}S^{(1,0,0)}_{i} + \frac{5}{3n}\left[R^{(1,0)}_{i}(x,\hat{x})\left(R^{(2,0)}(x,\check{x}) - \frac{3KT}{m}R^{(0,0)}(x,\check{x})\right) + R^{(1,0)}_{i}(x,\check{x})\left(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,0)}(x,\hat{x})\right)\right]$$

$$S^{(3,1,0)(0)}_{i,j} = \frac{5KT}{m}S^{(1,1,0)}_{i,j} + \frac{5}{3n}\left[R^{(1,2)}_{i,j}(x,\hat{x})\left(R^{(2,0)}(x,\check{x}) - \frac{3KT}{m}R^{(0,0)}(x,\check{x})\right) + R^{(1,0)}_{i}(x,\check{x})\left(R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m}R^{(0,1)}_{j}(x,\hat{x})\right)\right]$$

$$S^{(3,1,1)(0)}_{i,j,k} = \frac{5KT}{m}S^{(1,1,1)}_{i,j,k} + \frac{5}{3n}\left[R^{(1,1)}_{i,j}(x,\hat{x})\left(R^{(2,1)}_{k}(x,\check{x}) - \frac{3KT}{m}R^{(0,1)}_{k}(x,\check{x})\right) + R^{(1,1)}_{i,k}(x,\check{x})\left(R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m}R^{(0,1)}_{j}(x,\hat{x})\right)\right]$$

$$S^{(3,2,1)(0)}_{i,\ j} = \frac{5KT}{m} S^{(1,2,1)}_{i,\ j}$$

$$+ \frac{5}{3n}\left[R^{(1,2)}_{i}(x,\hat{x})\Big(R^{(2,1)}_{j}(x,\check{x}) - \frac{3KT}{m} R^{(0,1)}_{j}(x,\check{x})\Big)\right.$$

$$\left. + R^{(1,1)}_{i,j}(x,\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\Big)\right]$$

$$S^{(3,2,0)(0)}_{i} = \frac{5KT}{m} S^{(1,2,0)}_{i}$$

$$+ \frac{5}{3n}\left[R^{(1,2)}_{i}(x,\hat{x})\Big(R^{(2,0)}(x,\check{x}) - \frac{3KT}{m} R^{(0,0)}(x,\check{x})\Big)\right.$$

$$\left. + R^{(1,0)}_{i}(x,\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\Big)\right]$$

$$S^{(3,2,2)(0)}_{i} = \frac{5KT}{m} S^{(1,2,2)}_{i}$$

$$+ \frac{5}{3n}\left[R^{(1,2)}_{i}(x,\hat{x})\Big(R^{(2,2)}(x,\check{x}) - \frac{3KT}{m} R^{(0,2)}(x,\check{x})\Big)\right.$$

$$\left. + R^{(1,2)}_{i}(x,\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\Big)\right]$$

Substituting the right-hand sides of the above equalities for the corresponding S's in the conservation equations, we get the following 125 approximate equations governing the evolution of 125 S's:

$$\frac{\Delta S^{(0,0,0)}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(0,0,0)} + \frac{\partial S^{(1,0,0)}_{k}}{\partial x_k} + \frac{\partial S^{(0,1,0)}_{k}}{\partial \hat{x}_k}$$

$$+ \frac{\partial S^{(0,0,1)}_{k}}{\partial \check{x}_k} = 0 \tag{114}$$

$$\frac{\Delta S^{(1,0,0)}_{i}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(1,0,0)}_{i} + \frac{Du_i}{Dt} S^{(0,0,0)}$$

$$+ S^{(1,0,0)}_{k} \frac{\partial u_i}{\partial x_k} + \frac{\partial S^{(1,1,0)}_{i,k}}{\partial \hat{x}_k} + \frac{\partial S^{(1,0,1)}_{i,\ k}}{\partial \check{x}_k}$$

$$+ \frac{\partial}{\partial x_k}\left\{ \frac{1}{n}\left[R^{(1,0)}_{i}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x}) + R^{(1,0)}_{k}(x,\hat{x}) R^{(1,0)}_{i}(x,\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial x_i}\left[S^{(2,0,0)} - \frac{2}{n} R^{(1,0)}_{k}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x})\right] = 0$$

$$(1 \leqslant i \leqslant 3) \tag{115}$$

$$\frac{\Delta S^{(1,1,0)}_{i,j}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(1,1,0)}_{i,j} + \frac{Du_i}{Dt} S^{(0,1,0)}_{j}$$

$$+ \frac{\hat{D}\hat{u}_j}{\hat{D}t} S^{(1,0,0)}_{i} + \frac{\partial u_i}{\partial x_k} S^{(1,1,0)}_{k,j} + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(1,1,0)}_{i,k}$$

$$+ \frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,1)}_{i,j}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x}) + R^{(1,1)}_{k,j}(x,\hat{x}) R^{(1,0)}_{i}(x,\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial x_i}\left[S^{(2,1,0)}_{j} - \frac{2}{n} R^{(1,1)}_{k,j}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x})\right]$$

$$+ \frac{\partial}{\partial \hat{x}_k}\left\{\frac{1}{\hat{n}}\left[R^{(1,1)}_{i,j}(x,\hat{x}) R^{(1,0)}_{k}(\hat{x},\check{x}) + R^{(1,1)}_{i,k}(x,\hat{x}) R^{(1,0)}_{j}(\hat{x},\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial \hat{x}_j}\left[S^{(1,2,0)}_{i} - \frac{2}{\hat{n}} R^{(1,1)}_{i,k}(x,\hat{x}) R^{(1,0)}_{k}(\hat{x},\check{x})\right] + \frac{\partial S^{(1,1,1)}_{i,j,k}}{\partial \check{x}_k} = 0$$

$$(1 \leqslant i,j \leqslant 3) \quad (116)$$

$$\frac{\Delta S^{(1,1,1)}_{i,j,l}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(1,1,1)}_{i,j,l} + \frac{Du_i}{Dt} S^{(0,1,1)}_{j,l}$$

$$+ \frac{\hat{D}\hat{u}_j}{\hat{D}t} S^{(1,0,1)}_{i,\ l} + \frac{\check{D}\check{u}_l}{\check{D}t} S^{(1,1,0)}_{i,j} + \frac{\partial u_i}{\partial x_k} S^{(1,1,1)}_{i,j,l} + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(1,1,1)}_{i,k,l}$$

$$+ \frac{\partial \check{u}_l}{\partial \check{x}_k} S^{(1,1,1)}_{i,j,k} + \frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,1)}_{i,j}(x,\hat{x}) R^{(1,1)}_{k,l}(x,\check{x})\right.\right.$$

$$\left.\left. + R^{(1,1)}_{k,j}(x,\hat{x}) R^{(1,1)}_{i,l}(x,\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial x_i}\left[S^{(2,1,1)}_{j,l} - \frac{2}{n} R^{(1,1)}_{k,j}(x,\hat{x}) R^{(1,1)}_{k,l}(x,\check{x})\right]$$

$$+ \frac{\partial}{\partial \hat{x}_k}\left\{\frac{1}{\hat{n}}\left[R^{(1,1)}_{i,k}(x,\hat{x}) R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(1,1)}_{i,j}(x,\hat{x}) R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial \hat{x}_j}\left[S^{(1,2,1)}_{i,\ l} - \frac{2}{\hat{n}} R^{(1,1)}_{i,k}(x,\hat{x}) R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]$$

$$+ \frac{\partial}{\partial \check{x}_k}\left\{\frac{1}{\check{n}}\left[R^{(1,1)}_{i,k}(x,\check{x}) R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(1,1)}_{i,l}(x,\check{x}) R^{(1,1)}_{j,k}(\hat{x},\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial \check{x}_l}\left[S^{(1,1,2)}_{i,j} - \frac{2}{\check{n}} R^{(1,1)}_{i,k}(x,\check{x}) R^{(1,1)}_{j,k}(\hat{x},\check{x})\right] = 0$$

$$(1 \leqslant i,j,l \leqslant 3) \quad (117)$$

$$
\begin{aligned}
&\frac{\Delta S^{(2,0,0)}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(2,0,0)} + 2 S^{(1,0,0)}_{k} \frac{D u_k}{D t} \\
&\quad + 2 \frac{\partial u_k}{\partial x_s} \frac{1}{n} \left[R^{(1,0)}_{k}(x,\hat{x}) R^{(1,0)}_{s}(x,\check{x}) + R^{(1,0)}_{s}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x}) \right] \\
&\quad + \frac{2}{3} \left(\frac{\partial u_k}{\partial x_k}\right) \left[S^{(2,0,0)} - \frac{2}{n} R^{(1,0)}_{k}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x}) \right] \\
&\quad + \frac{5K}{m} \frac{\partial}{\partial x_k} \left(T S^{(1,0,0)}_{k} \right) \\
&\quad + \frac{5}{3} \frac{\partial}{\partial x_k} \left\{ \frac{1}{n} \left[R^{(1,0)}_{k}(x,\hat{x}) \left(R^{(2,0)}(x,\check{x}) - \frac{3KT}{m} R^{(0,0)}(x,\check{x}) \right) \right.\right. \\
&\qquad\qquad \left.\left. + R^{(1,0)}_{k}(x,\check{x}) \left(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,0)}(x,\hat{x}) \right) \right] \right\} \\
&\quad + \frac{\partial S^{(2,1,0)}_{k}}{\partial \hat{x}_k} + \frac{\partial S^{(2,0,1)}_{k}}{\partial \check{x}_k} = 0 \qquad (118)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\Delta S^{(2,1,0)}_{j}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(2,1,0)}_{j} + 2 S^{(1,1,0)}_{k,j} \frac{D u_k}{D t} \\
&\quad + S^{(2,0,0)} \frac{\hat{D} \hat{u}_j}{\hat{D} t} + \frac{2}{n} \frac{\partial u_k}{\partial x_s} \left[R^{(1,1)}_{k,j}(x,\hat{x}) R^{(1,0)}_{s}(x,\check{x}) \right. \\
&\qquad\qquad \left. + R^{(1,1)}_{s,j}(x,\hat{x}) R^{(1,0)}_{k}(x,\check{x}) \right] \\
&\quad + \frac{2}{3} \left(\frac{\partial u_k}{\partial x_k}\right) \left[S^{(2,1,0)}_{j} - \frac{2}{n} R^{(1,1)}_{l,j}(x,\hat{x}) R^{(1,0)}_{l}(x,\check{x}) \right] \\
&\quad + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(2,1,0)}_{k} + \frac{5K}{m} \frac{\partial}{\partial x_k} \left(T S^{(1,1,0)}_{k,j} \right) \\
&\quad + \frac{5}{3} \frac{\partial}{\partial x_k} \left\{ \frac{1}{n} \left[R^{(1,1)}_{k,j}(x,\hat{x}) \left(R^{(2,0)}(x,\check{x}) - \frac{3KT}{m} R^{(0,0)}(x,\check{x}) \right) \right.\right. \\
&\qquad\qquad \left.\left. + R^{(1,0)}_{k}(x,\check{x}) \left(R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m} R^{(0,1)}_{j}(x,\hat{x}) \right) \right] \right\} \\
&\quad + \frac{\partial}{\partial \hat{x}_k} \left\{ \frac{1}{n} \left[R^{(2,1)}_{k}(x,\hat{x}) R^{(1,0)}_{j}(\hat{x},\check{x}) + R^{(2,1)}_{j}(x,\hat{x}) R^{(1,0)}_{k}(\hat{x},\check{x}) \right] \right\} \\
&\quad + \frac{1}{3} \frac{\partial}{\partial \hat{x}_j} \left[S^{(2,2,0)} - \frac{2}{\hat{n}} R^{(2,1)}_{k}(x,\hat{x}) R^{(1,0)}_{k}(\hat{x},\check{x}) \right] + \frac{\partial S^{(2,1,1)}_{j,k}}{\partial \check{x}_k} = 0 \\
&\qquad\qquad (1 \leqslant j \leqslant 3) \qquad (119)
\end{aligned}
$$

$$\frac{\Delta S^{(2,1,1)}_{j,l}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(2,1,1)}_{j,l} + 2\frac{Du_k}{Dt} S^{(1,1,1)}_{k,j,l}$$

$$+ \frac{\hat{D}\hat{u}_j}{\hat{D}t} S^{(2,0,1)}_{l} + \frac{\check{D}\check{u}_l}{\check{D}t} S^{(2,1,0)}_{j} + \frac{2}{n}\frac{\partial u_k}{\partial x_s}\left[R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,1)}_{s,l}(x,\check{x})\right.$$

$$\left. + R^{(1,1)}_{s,j}(x,\hat{x})R^{(1,1)}_{k,l}(x,\check{x})\right]$$

$$+ \frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\left[S^{(2,1,1)}_{j,l} - \frac{1}{n}R^{(1,1)}_{m,j}(x,\hat{x})R^{(1,1)}_{m,l}(x,\check{x})\right]$$

$$+ \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(2,1,1)}_{k,l} + \frac{\partial \check{u}_l}{\partial \check{x}_k} S^{(2,1,1)}_{j,k} + \frac{5K}{m}\frac{\partial}{\partial x_k}\left(TS^{(1,1,1)}_{k,j,l}\right)$$

$$+ \frac{5}{3}\frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,1)}_{k,j}(x,\hat{x})\left(R^{(2,1)}_{l}(x,\check{x}) - \frac{3KT}{m}R^{(0,1)}_{l}(x,\check{x})\right)\right.\right.$$

$$\left.\left. + R^{(1,1)}_{k,l}(x,\check{x})\left(R^{(2,1)}_{k}(x,\hat{x}) - \frac{3KT}{m}R^{(0,1)}_{k}(x,\hat{x})\right)\right]\right\}$$

$$+ \frac{\partial}{\partial \hat{x}_k}\left\{\frac{1}{\hat{n}}\left[R^{(2,1)}_{k}(x,\hat{x})R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(2,1)}_{j}(x,\hat{x})R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial \hat{x}_j}\left[S^{(2,2,1)}_{l} - \frac{2}{\hat{n}}R^{(2,1)}_{k}(x,\hat{x})R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]$$

$$+ \frac{\partial}{\partial \check{x}_k}\left\{\frac{1}{\check{n}}\left[R^{(2,1)}_{k}(x,\check{x})R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(2,1)}_{l}(x,\check{x})R^{(1,1)}_{j,k}(\hat{x},\check{x})\right]\right\}$$

$$+ \frac{1}{3}\frac{\partial}{\partial \check{x}_l}\left[S^{(2,1,2)}_{j} - \frac{2}{\check{n}}R^{(2,1)}_{k}(x,\check{x})R^{(1,1)}_{j,k}(\hat{x},\check{x})\right] = 0$$

$$(1 \leqslant j,l \leqslant 3) \qquad (120)$$

$$\frac{\Delta S^{(2,2,1)}_{l}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right) S^{(2,2,1)}_{l} + 2\frac{Du_k}{Dt} S^{(1,2,1)}_{k,\ l}$$

$$+ 2\frac{\hat{D}\hat{u}_k}{\hat{D}t} S^{(2,1,1)}_{k,l} + \frac{\check{D}\check{u}_l}{\check{D}t} S^{(2,2,0)} + \frac{2}{n}\frac{\partial u_k}{\partial x_s}$$

$$\times\left[R^{(1,2)}_{k}(x,\hat{x})R^{(1,1)}_{s,l}(x,\check{x}) + R^{(1,2)}_{s}(x,\hat{x})R^{(1,1)}_{k,l}(x,\check{x})\right] + \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\times\left[S^{(2,2,1)}_{l} - \frac{1}{n}R^{(1,2)}_{m}(x,\hat{x})R^{(1,1)}_{m,l}(x,\check{x})\right] =$$

$$+\frac{2}{\hat{n}}\frac{\partial \hat{u}_k}{\partial \hat{x}_s}\left[R^{(2,1)}_{k}(x,\hat{x})R^{(1,1)}_{s,l}(\hat{x},\check{x})+R^{(2,1)}_{s}(x,\hat{x})R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]$$

$$+\frac{2}{3}\left(\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)\left[S^{(2,2,1)}_{l}-\frac{2}{\hat{n}}R^{(2,1)}_{m}(x,\hat{x})R^{(1,1)}_{m,l}(\hat{x},\check{x})\right]$$

$$+\frac{\partial \check{u}_l}{\partial \check{x}_k}S^{(2,2,1)}_{k}+\frac{5K}{m}\frac{\partial}{\partial x_k}\left(TS^{(1,2,1)}_{k,\ l}\right)$$

$$+\frac{5}{3}\frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,2)}_{k}(x,\hat{x})\left(R^{(2,1)}_{l}(x,\check{x})-\frac{3KT}{m}R^{(0,1)}_{l}(x,\check{x})\right)+R^{(1,1)}_{k,l}(x,\check{x})\right.\right.$$
$$\left.\left.\times\left(R^{(2,2)}(x,\hat{x})-\frac{3KT}{m}R^{(0,2)}(x,\hat{x})\right)\right]\right\}+\frac{5K}{m}\frac{\partial}{\partial \hat{x}_k}\left(\hat{T}S^{(2,1,1)}_{k,l}\right)$$

$$+\frac{5}{3}\frac{\partial}{\partial \hat{x}_k}\left\{\frac{1}{\hat{n}}\left[R^{(2,1)}_{k}(x,\hat{x})\left(R^{(2,1)}_{l}(\hat{x},\check{x})-\frac{3K\hat{T}}{m}R^{(0,1)}_{l}(\hat{x},\check{x})\right)\right.\right.$$
$$\left.\left.+R^{(1,1)}_{k,l}(\hat{x},\check{x})\left(R^{(2,2)}(x,\hat{x})-\frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\right)\right]\right\}$$

$$+\frac{\partial}{\partial \check{x}_k}\left\{\frac{1}{\check{n}}\left[R^{(2,1)}_{k}(x,\check{x})R^{(2,1)}_{l}(\hat{x},\check{x})+R^{(2,1)}_{l}(x,\check{x})R^{(2,1)}_{k}(\hat{x},\check{x})\right]\right\}$$

$$+\frac{1}{3}\frac{\partial}{\partial \check{x}_l}\left[S^{(2,2,2)}-\frac{2}{\check{n}}R^{(2,1)}_{k}(x,\check{x})R^{(2,1)}_{k}(\hat{x},\check{x})\right]=0\qquad(1\leqslant l\leqslant 3)\tag{121}$$

$$\frac{\Delta S^{(2,2,0)}}{\Delta t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}+\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,2,0)}+2\frac{Du_k}{Dt}S^{(1,2,0)}_{k}+2\frac{\hat{D}\hat{u}_k}{\hat{D}t}S^{(2,1,0)}_{k}$$

$$+\frac{2}{n}\frac{\partial u_k}{\partial x_s}\left[R^{(1,2)}_{k}(x,\hat{x})R^{(1,0)}_{s}(x,\check{x})+R^{(1,2)}_{s}(x,\hat{x})R^{(1,0)}_{k}(x,\check{x})\right]$$

$$+\frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\left[S^{(2,2,0)}-\frac{2}{n}R^{(1,2)}_{l}(x,\hat{x})R^{(1,0)}_{l}(x,\check{x})\right]$$

$$+\frac{2}{\hat{n}}\frac{\partial \hat{u}_k}{\partial \hat{x}_s}\left[R^{(2,1)}_{k}(x,\hat{x})R^{(1,0)}_{s}(\hat{x},\check{x})+R^{(2,1)}_{s}(x,\hat{x})R^{(1,0)}_{k}(\hat{x},\check{x})\right]$$

$$+\frac{2}{3}\left(\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)\left[S^{(2,2,0)}-\frac{2}{\hat{n}}R^{(2,1)}_{l}(x,\hat{x})R^{(1,0)}_{l}(\hat{x},\check{x})\right]$$

$$+\frac{5K}{m}\frac{\partial}{\partial x_k}\left(TS^{(1,2,0)}_{k}\right)$$

$$+\frac{5}{3}\frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,2)}_{k}(x,\hat{x})\left(R^{(2,0)}(x,\check{x})-\frac{3KT}{m}R^{(0,0)}(x,\check{x})\right)=\right.\right.$$

$$+ R^{(1,0)}_{k}(x,\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,2)}(x,\hat{x})\Big)\Big]\Big\} + \frac{5K}{m}\frac{\partial}{\partial \hat{x}_k}\big(\hat{T}S^{(2,1,0)}_{k}\big)$$

$$+ \frac{5}{3}\frac{\partial}{\partial \hat{x}_k}\Big\{\frac{1}{\hat{n}}\Big[R^{(2,1)}_{k}(x,\hat{x})\Big(R^{(2,0)}(\hat{x},\check{x}) - \frac{3K\hat{T}}{m}R^{(0,0)}(\hat{x},\check{x})\Big)$$

$$+ R^{(1,0)}_{k}(\hat{x},\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\Big)\Big]\Big\}$$

$$+ \frac{\partial S^{(2,2,1)}_{k}}{\partial \check{x}_k} = 0 \tag{122}$$

$$\frac{\Delta S^{(2,2,2)}}{\Delta t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} + \frac{\partial \check{u}_k}{\partial \check{x}_k}\right)S^{(2,2,2)} + 2\frac{Du_k}{Dt}S^{(1,2,2)}_{k}$$

$$+ 2\frac{\hat{D}\hat{u}_k}{\hat{D}t}S^{(2,1,2)}_{k} + 2\frac{\check{D}\check{u}_k}{\check{D}t}S^{(2,2,1)}_{k}$$

$$+ \frac{2}{n}\frac{\partial u_k}{\partial x_s}\Big[R^{(1,2)}_{k}(x,\hat{x})R^{(1,2)}_{s}(x,\check{x}) + R^{(1,2)}_{s}(x,\hat{x})R^{(1,2)}_{k}(x,\check{x})\Big]$$

$$+ \frac{2}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\Big[S^{(2,2,2)} - \frac{2}{n}R^{(1,2)}_{l}(x,\hat{x})R^{(1,2)}_{l}(x,\check{x})\Big]$$

$$+ \frac{2}{\hat{n}}\frac{\partial \hat{u}_k}{\partial \hat{x}_s}\Big[R^{(2,1)}_{k}(x,\hat{x})R^{(1,2)}_{s}(\hat{x},\check{x}) + R^{(2,1)}_{s}(x,\hat{x})R^{(1,2)}_{k}(\hat{x},\check{x})\Big]$$

$$+ \frac{2}{3}\left(\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)\Big[S^{(2,2,2)} - \frac{2}{\hat{n}}R^{(2,1)}_{l}(x,\hat{x})R^{(1,2)}_{l}(\hat{x},\check{x})\Big]$$

$$+ \frac{2}{\check{n}}\frac{\partial \check{u}_k}{\partial \check{x}_s}\Big[R^{(2,1)}_{k}(x,\check{x})R^{(2,1)}_{s}(\hat{x},\check{x}) + R^{(2,1)}_{s}(x,\check{x})R^{(2,1)}_{k}(\hat{x},\check{x})\Big]$$

$$+ \frac{2}{3}\left(\frac{\partial \check{u}_k}{\partial \check{x}_k}\right)\Big[S^{(2,2,2)} - \frac{2}{\check{n}}R^{(2,1)}_{l}(x,\check{x})R^{(2,1)}_{l}(\hat{x},\check{x})\Big]$$

$$+ \frac{5K}{m}\frac{\partial}{\partial x_k}\big(TS^{(1,2,2)}_{k}\big)$$

$$+ \frac{5}{3}\frac{\partial}{\partial x_k}\Big\{\frac{1}{n}\Big[R^{(1,2)}_{k}(x,\hat{x})\Big(R^{(2,2)}(x,\check{x}) - \frac{3KT}{m}R^{(0,2)}(x,\check{x})\Big)$$

$$+ R^{(1,2)}_{k}(x,\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,2)}(x,\hat{x})\Big)\Big]\Big\}$$

$$+ \frac{5K}{m}\frac{\partial}{\partial \hat{x}_k}\big(\hat{T}S^{(2,1,2)}_{k}\big)$$

$$+ \frac{5}{3}\frac{\partial}{\partial \hat{x}_k}\Big\{\frac{1}{\hat{n}}\Big[R^{(2,1)}_{k}(x,\hat{x})\Big(R^{(2,2)}(\hat{x},\check{x}) - \frac{3K\hat{T}}{m}R^{(0,2)}(\hat{x},\check{x})\Big) =$$

$$+ R^{(1,2)}_{\ \ k}(\hat{x},\check{x})\Big(R^{(2,2)}(x,\hat{x}) - \frac{3K\hat{T}}{m} R^{(2,0)}(x,\hat{x})\Big)\Big]\Big\} + \frac{5K}{m}\frac{\partial}{\partial \check{x}_k}\big(\check{T}S^{(2,2,1)}_{\quad\ \ k}\big)$$

$$+ \frac{5}{3}\frac{\partial}{\partial \check{x}_k}\Big\{\frac{1}{\check{n}}\Big[R^{(2,1)}_{\ \ k}(x,\check{x})\Big(R^{(2,2)}(\hat{x},\check{x}) - \frac{3K\check{T}}{m} R^{(2,0)}(\hat{x},\check{x})\Big)$$

$$+ R^{(2,1)}_{\ \ k}(\hat{x},\check{x})\Big(R^{(2,2)}(x,\check{x}) - \frac{3K\check{T}}{m} R^{(2,0)}(x,\check{x})\Big)\Big]\Big\} = 0 \tag{123}$$

Equations (114)–(123) represent 125 equations which, together with the 30 equations represented by (53)–(61), form a system of approximate equations governing the motion of turbulent flows. We call this system the approximate equation system of order $1\frac{1}{3}$ for turbulent flows. Since the 125 S's do not appear in (53)–(61), we can solve equations (53)–(61) independently. This fact makes the approximate system of equations of order $1\frac{1}{3}$ mathematically insignificant.

On account of (113) and the result for the zeroth-order approximation of the R's, we have

$$R^{(2,0)(0)}_{ik} + R^{(2,0)(1)}_{ik}$$

$$= \frac{1}{n^2}\Big\{nS^{(1,0,1)}_{\ i,\ \ k}(x,\hat{x},x) - R^{(1,1)}_{\ i,k}(x,x)R^{(0,0)}(x,\hat{x})$$

$$- R^{(1,0)}_{\ i}(x,\hat{x})R^{(1,0)}_{\ k}(x,x) - R^{(1,0)}_{\ k}(x,\hat{x})R^{(1,0)}_{\ i}(x,x)$$

$$- \frac{n}{18B_1}\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big)\Big[R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,0)}(x,\hat{x})\Big]$$

$$- \frac{n^2KT}{6B_1 m}\Big[\frac{\partial}{\partial x_k}\Big(\frac{1}{n}R^{(1,0)}_{\ i}(x,\hat{x})\Big) + \frac{\partial}{\partial x_i}\Big(\frac{1}{n}R^{(1,0)}_{\ k}(x,\hat{x})\Big)\Big]$$

$$+ \frac{\delta_{ik}}{3}\Big[n^2R^{(2,0)}(x,\hat{x}) - \Big(nS^{(1,0,1)}_{\ l,\ \ l}(x,\hat{x},x) - R^{(0,0)}(x,\hat{x})R^{(1,1)}_{\ l,l}(x,x)$$

$$- 2R^{(1,0)}_{\ l}(x,\hat{x})R^{(1,0)}_{\ l}(x,x)$$

$$- \frac{n}{9B_1}\Big(\frac{\partial u_l}{\partial x_l}\Big)\Big(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,0)}(x,\hat{x})\Big)$$

$$- \frac{n^2KT}{3B_1 m}\frac{\partial}{\partial x_l}\Big(\frac{1}{n}R^{(1,0)}_{\ l}(x,\hat{x})\Big)\Big)\Big]\Big\} =$$

$$
\begin{aligned}
R_{ik,j}^{(2,1)(0)} + R_{ik,j}^{(2,1)(1)}
= \frac{1}{n^2} \Bigg\{ & nS^{\binom{1,1,1}{i,j,k}}(x,\hat{x},x) - R^{\binom{1,1}{i,k}}(x,x) R^{\binom{0,1}{j}}(x,\hat{x}) \\
& - R^{\binom{1,1}{i,j}}(x,\hat{x}) R^{\binom{1,0}{k}}(x,x) - R^{\binom{1,1}{k,j}}(x,\hat{x}) R^{\binom{1,0}{i}}(x,x) \\
& - \frac{n}{18B_1}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left[R^{\binom{2,1}{j}}(x,\hat{x}) - \frac{3KT}{m} R^{\binom{0,1}{j}}(x,\hat{x})\right] \\
& - \frac{n^2KT}{6B_1m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n} R^{\binom{1,1}{i,j}}(x,\hat{x})\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{n} R^{\binom{1,1}{k,j}}(x,\hat{x})\right)\right] \\
& + \frac{\delta_{ik}}{3}\Bigg[n^2 R^{\binom{2,1}{j}}(x,\hat{x}) - \Bigg(nS^{\binom{1,1,1}{l,j,l}}(x,\hat{x},x) \\
& \qquad - R^{\binom{0,1}{j}}(x,\hat{x}) R^{\binom{1,1}{l,l}}(x,x) - 2R^{\binom{1,1}{l,j}}(x,\hat{x}) R^{\binom{1,0}{l}}(x,x) \\
& \qquad - \frac{n}{9B_1}\left(R^{\binom{2,1}{j}}(x,\hat{x}) - \frac{3KT}{m} R^{\binom{0,1}{j}}(x,\hat{x})\right)\left(\frac{\partial u_l}{\partial x_l}\right) \\
& \qquad - \frac{n^2KT}{3B_1m}\frac{\partial}{\partial x_l}\left(\frac{1}{n} R^{\binom{1,1}{l,j}}(x,\hat{x})\right)\Bigg)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
R_{ik}^{(2,2)(0)} + R_{ik}^{(2,2)(1)}
= \frac{1}{n^2} \Bigg\{ & nS^{\binom{1,2,1}{i,\ ,k}}(x,\hat{x},x) - R^{\binom{1,1}{i,k}}(x,x) R^{(0,2)}(x,\hat{x}) \\
& - R^{\binom{1,2}{i}}(x,\hat{x}) R^{\binom{1,0}{k}}(x,x) - R^{\binom{1,2}{k}}(x,\hat{x}) R^{\binom{1,0}{i}}(x,x) \\
& - \frac{n}{18B_1}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\right)\left[R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\right] \\
& - \frac{n^2KT}{6B_1m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n} R^{\binom{1,2}{i}}(x,\hat{x})\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{n} R^{\binom{1,2}{k}}(x,\hat{x})\right)\right] \\
& + \frac{\delta_{ik}}{3}\Bigg[n^2 R^{(2,2)}(x,\hat{x}) - \Bigg(nS^{\binom{1,2,1}{l,\ ,l}}(x,\hat{x},x) - R^{(0,2)}(x,\hat{x}) R^{\binom{1,1}{l,l}}(x,x) \\
& \qquad - 2R^{\binom{1,2}{l}}(x,\hat{x}) R^{\binom{1,0}{l}}(x,x) =
\end{aligned}
$$

$$- \frac{n}{9B_1}\left(\frac{\partial u_l}{\partial x_l}\right)\left(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,2)}(x,\hat{x})\right)$$

$$- \frac{n^2KT}{3B_1 m}\frac{\partial}{\partial x_l}\left(\frac{1}{n}R^{(1,2)}_{l}(x,\hat{x})\right)\Bigg)\Bigg]\Bigg\}$$

$$R^{(3,0)(0)}_{i} + R^{(3,0)(1)}_{i}$$

$$= \frac{1}{n^2}\Bigg\{-\frac{4}{3}R^{(1,0)}_{k}(x,\hat{x})R^{(1,1)}_{k,i}(x,x) - \frac{2}{3}R^{(1,0)}_{i}(x,\hat{x})R^{(1,1)}_{k,k}(x,x)$$

$$+ \frac{5}{3}R^{(1,0)}_{i}(x,\hat{x})\left[\frac{3KT}{m}R^{(0,0)}(x,x) - R^{(0,2)}(x,x)\right]$$

$$+ \frac{5}{3}R^{(1,0)}_{i}(x,x)\left[\frac{3KT}{m}R^{(0,0)}(x,\hat{x}) - R^{(2,0)}(x,\hat{x})\right]$$

$$+ \frac{5}{3}R^{(0,0)}(x,\hat{x})\left[\frac{3KT}{m}R^{(1,0)}_{i}(x,x) - R^{(1,2)}_{i}(x,x)\right]$$

$$+ \frac{5n}{3}\left[S^{(1,0,2)}_{i}(x,\hat{x},x) - \frac{3KT}{m}S^{(1,0,0)}_{i}(x,\hat{x},x)\right]$$

$$+ \frac{2KTn}{9B_1 m}\left(\frac{\partial u_k}{\partial x_k}\right)R^{(1,0)}_{i}(x,\hat{x}) - \frac{KTn}{3B_1 m}R^{(1,0)}_{k}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$- \frac{5Kn}{6B_1 m}\left(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,0)}(x,\hat{x})\right)\frac{\partial T}{\partial x_i}$$

$$- \frac{5Kn^2T^2}{12B_1 m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT}R^{(2,0)}(x,\hat{x}) - \frac{3K}{mn}R^{(0,0)}(x,\hat{x})\right]\Bigg\}$$

$$+ \frac{5KT}{m}R^{(1,0)}_{i}(x,\hat{x})$$

$$R^{(3,1)(0)}_{i,j} + R^{(3,1)(1)}_{i,j}$$

$$= \frac{1}{n^2}\Bigg\{-\frac{4}{3}R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,1)}_{k,i}(x,x) - \frac{2}{3}R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,1)}_{k,k}(x,x)$$

$$+ \frac{5}{3}R^{(1,1)}_{i,j}(x,\hat{x})\left[\frac{3KT}{m}R^{(0,0)}(x,x) - R^{(0,2)}(x,x)\right]$$

$$+ \frac{5}{3}R^{(1,0)}_{i}(x,x)\left[\frac{3KT}{m}R^{(0,1)}_{j}(x,\hat{x}) - R^{(2,1)}_{j}(x,\hat{x})\right] =$$

$$+ \frac{5}{3} R^{(0,1)}_{j}(x,\hat{x})\left[\frac{3KT}{m} R^{(1,0)}_{i}(x,x) - R^{(1,2)}_{i}(x,x)\right]$$

$$+ \frac{5n}{3}\left[S^{(1,1,2)}_{i,j}(x,\hat{x},x) - \frac{3KT}{m} S^{(1,1,0)}_{i,j}(x,\hat{x},x)\right]$$

$$+ \frac{2KTn}{9B_1 m}\left(\frac{\partial u_k}{\partial x_k}\right) R^{(1,1)}_{i,j}(x,\hat{x}) - \frac{KTn}{3B_1 m} R^{(1,1)}_{k,j}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$- \frac{5Kn}{6B_1 m}\left(R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m} R^{(0,1)}_{j}(x,\hat{x})\right)\frac{\partial T}{\partial x_i}$$

$$- \frac{5Kn^2T^2}{12B_1 m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT} R^{(2,1)}_{j}(x,\hat{x}) - \frac{3K}{mn} R^{(0,1)}_{j}(x,\hat{x})\right]\Bigg\}$$

$$+ \frac{5KT}{m} R^{(1,1)}_{i,j}(x,\hat{x})$$

$$R^{(3,2)(0)}_{i} + R^{(3,2)(1)}_{i}$$

$$= \frac{1}{n^2}\Bigg\{ -\frac{4}{3} R^{(1,2)}_{k}(x,\hat{x}) R^{(1,1)}_{k,i}(x,x) - \frac{2}{3} R^{(1,2)}_{i}(x,\hat{x}) R^{(1,1)}_{k,k}(x,x)$$

$$+ \frac{5}{3} R^{(1,2)}_{i}(x,\hat{x})\left[\frac{3KT}{m} R^{(0,0)}(x,x) - R^{(0,2)}(x,x)\right]$$

$$+ \frac{5}{3} R^{(1,0)}_{i}(x,x)\left[\frac{3KT}{m} R^{(0,2)}(x,\hat{x}) - R^{(2,2)}(x,\hat{x})\right]$$

$$+ \frac{5}{3} R^{(0,2)}(x,\hat{x})\left[\frac{3KT}{m} R^{(1,0)}_{i}(x,x) - R^{(1,2)}_{i}(x,x)\right]$$

$$+ \frac{5n}{3}\left[S^{(1,2,2)}_{i}(x,\hat{x},x) - \frac{3KT}{m} S^{(1,2,0)}_{i}(x,\hat{x},x)\right]$$

$$+ \frac{2KTn}{9B_1 m}\left(\frac{\partial u_k}{\partial x_k}\right) R^{(1,2)}_{i}(x,\hat{x}) - \frac{KTn}{3B_1 m} R^{(1,2)}_{k}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$$- \frac{5Kn}{6B_1 m}\left(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\right)\frac{\partial T}{\partial x_i}$$

$$- \frac{5Kn^2T^2}{12B_1 m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT} R^{(2,2)}(x,\hat{x}) - \frac{3K}{mn} R^{(0,2)}(x,\hat{x})\right]\Bigg\}$$

$$+ \frac{5KT}{m} R^{(1,2)}_{i}(x,\hat{x})$$

Now we can make a new closure of the conservation equations (20)–(25) as follows:

$$\frac{\mathscr{D} R^{(0,0)}}{\mathscr{D} t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) R^{(0,0)}+\frac{\partial R^{(1,0)}_{k}}{\partial x_k}+\frac{\partial R^{(0,1)}_{k}}{\partial \hat{x}_k}=0 \qquad (124)$$

$$\frac{\mathscr{D} R^{(1,0)}_{i}}{\mathscr{D} t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) R^{(1,0)}_{i}+\frac{D u_i}{D t} R^{(0,0)}+R^{(1,0)}_{k} \frac{\partial u_i}{\partial x_k}$$

$$+\frac{\partial}{\partial x_k}\left\{\frac{1}{n} S^{(1,0,1)}_{i,\ k}(x, \hat{x}, x)-\frac{1}{n^2} R^{(1,1)}_{i,k}(x, x) R^{(0,0)}(x, \hat{x})\right.$$

$$-\frac{1}{n^2} R^{(1,0)}_{i}(x, \hat{x}) R^{(1,0)}_{k}(x, x)-\frac{1}{n^2} R^{(1,0)}_{k}(x, \hat{x}) R^{(1,0)}_{i}(x, x)$$

$$-\frac{1}{18 B_1 n}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\left[R^{(2,0)}(x, \hat{x})-\frac{3 K T}{m} R^{(0,0)}(x, \hat{x})\right]$$

$$\left.-\frac{K T}{6 B_1 m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n} R^{(1,0)}_{i}(x, \hat{x})\right)+\frac{\partial}{\partial x_i}\left(\frac{1}{n} R^{(1,0)}_{k}(x, \hat{x})\right)\right]\right\}$$

$$+\frac{1}{3} \frac{\partial}{\partial x_i}\left\{R^{(2,0)}(x, \hat{x})-\left[\frac{1}{n} S^{(1,0,1)}_{l,\ \ l}(x, \hat{x}, x)-\frac{R^{(0,0)}(x, \hat{x})}{n^2}\right.\right.$$

$$\times R^{(1,1)}_{l,l}(x, x)-\frac{2}{n^2} R^{(1,0)}_{l}(x, \hat{x}) R^{(1,0)}_{l}(x, x)-\frac{1}{9 B_1 n}$$

$$\times\left(\frac{\partial u_l}{\partial x_l}\right)\left(R^{(2,0)}(x, \hat{x})-\frac{3 K T}{m} R^{(0,0)}(x, \hat{x})\right)-\frac{K T}{3 B_1 m} \frac{\partial}{\partial x_l}$$

$$\left.\left.\times\left(\frac{1}{n} R^{(1,0)}_{l}(x, \hat{x})\right)\right]\right\}+\frac{\partial R^{(1,1)}_{i,k}}{\partial \hat{x}_k}=0 \qquad (1 \leqslant i \leqslant 3) \qquad (125)$$

$$\frac{\mathscr{D} R^{(1,1)}_{i,j}}{\mathscr{D} t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) R^{(1,1)}_{i,j}+\frac{D u_i}{D t} R^{(0,1)}_{j}+\frac{\hat{D} \hat{u}_j}{\hat{D} t} R^{(1,0)}_{i}$$

$$+R^{(1,1)}_{k,j} \frac{\partial u_i}{\partial x_k}+R^{(1,1)}_{i,k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k}+\frac{\partial}{\partial x_k}\left\{\frac{1}{n} S^{(1,1,1)}_{i,j,k}(x, \hat{x}, x)\right.$$

$$-\frac{1}{n^2} R^{(1,1)}_{i,k}(x, x) R^{(0,1)}_{j}(x, \hat{x})-\frac{1}{n^2} R^{(1,1)}_{i,j}(x, \hat{x}) R^{(1,0)}_{k}(x, x)$$

$$-\frac{1}{n^2} R^{(1,1)}_{k,j}(x, \hat{x}) R^{(1,0)}_{i}(x, x)-\frac{1}{18 B_1 n}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)=$$

$$\times\left[R^{(2,1)}_{\ j}(x,\hat{x})-\frac{3KT}{m}R^{(0,1)}_{\ j}(x,\hat{x})\right]$$

$$-\frac{KT}{6B_1m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n}R^{(1,1)}_{i,j}(x,\hat{x})\right)+\frac{\partial}{\partial x_i}\left(\frac{1}{n}R^{(1,1)}_{k,j}(x,\hat{x})\right)\right]\Bigg\}$$

$$+\frac{1}{3}\frac{\partial}{\partial x_i}\Bigg\{R^{(2,1)}_{\ j}(x,\hat{x})-\Bigg[\frac{1}{n}S^{(1,1,1)}_{l,j,l}(x,\hat{x},x)-\frac{1}{n^2}R^{(0,1)}_{\ j}(x,\hat{x})R^{(1,1)}_{l,l}(x,x)$$

$$-\frac{2}{n^2}R^{(1,1)}_{l,j}(x,\hat{x})R^{(1,0)}_{l}(x,x)-\frac{1}{9B_1n}\left(\frac{\partial u_l}{\partial x_l}\right)$$

$$\times\left(R^{(2,1)}_{\ j}(x,\hat{x})-\frac{3KT}{m}R^{(0,1)}_{\ j}(x,\hat{x})\right)-\frac{KT}{3B_1m}\frac{\partial}{\partial x_l}\left(\frac{1}{n}R^{(1,1)}_{l,j}(x,\hat{x})\right)\Bigg]\Bigg\}$$

$$+\frac{\partial}{\partial\hat{x}_k}\Bigg\{\frac{1}{\hat{n}}S^{(1,1,1)}_{i,j,k}(x,\hat{x},\hat{x})-\frac{1}{\hat{n}^2}R^{(1,1)}_{k,j}(\hat{x},\hat{x})R^{(1,0)}_{i}(x,\hat{x})$$

$$-\frac{1}{\hat{n}^2}R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,0)}_{k}(\hat{x},\hat{x})-\frac{1}{\hat{n}^2}R^{(1,1)}_{i,k}(x,\hat{x})R^{(1,0)}_{j}(\hat{x},\hat{x})$$

$$-\frac{1}{18B_1\hat{n}}\left(\frac{\partial\hat{u}_j}{\partial\hat{x}_k}+\frac{\partial\hat{u}_k}{\partial\hat{x}_j}\right)\left[R^{(1,2)}_{i}(x,\hat{x})-\frac{3K\hat{T}}{m}R^{(1,0)}_{i}(x,\hat{x})\right]$$

$$-\frac{K\hat{T}}{6B_1m}\left[\frac{\partial}{\partial\hat{x}_k}\left(\frac{1}{\hat{n}}R^{(1,1)}_{i,j}(x,\hat{x})\right)+\frac{\partial}{\partial\hat{x}_j}\left(\frac{1}{\hat{n}}R^{(1,1)}_{i,k}(x,\hat{x})\right)\right]\Bigg\}$$

$$+\frac{1}{3}\frac{\partial}{\partial\hat{x}_j}\Bigg\{R^{(1,2)}_{i}(x,\hat{x})-\Bigg[\frac{1}{\hat{n}}S^{(1,1,1)}_{i,l,l}(x,\hat{x},\hat{x})$$

$$-\frac{1}{\hat{n}^2}R^{(1,0)}_{i}(x,\hat{x})R^{(1,1)}_{l,l}(\hat{x},\hat{x})-\frac{2}{\hat{n}^2}R^{(1,1)}_{i,l}(x,\hat{x})R^{(1,0)}_{l}(\hat{x},\hat{x})$$

$$-\frac{1}{9B_1\hat{n}}\left(\frac{\partial\hat{u}_l}{\partial\hat{x}_l}\right)\left(R^{(1,2)}_{i}(x,\hat{x})-\frac{3K\hat{T}}{m}R^{(1,0)}_{i}(x,\hat{x})\right)$$

$$-\frac{K\hat{T}}{3B_1m}\frac{\partial}{\partial\hat{x}_l}\left(\frac{1}{\hat{n}}R^{(1,1)}_{i,l}(x,\hat{x})\right)\Bigg]\Bigg\}=0\qquad(1\leqslant i,j\leqslant 3)\qquad(126)$$

$$\frac{\mathcal{D}R^{(2,0)}}{\mathcal{D}t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial\hat{u}_k}{\partial\hat{x}_k}\right)R^{(2,0)}+2\frac{Du_i}{Dt}R^{(1,0)}_{i}$$

$$+\frac{2}{n^2}\frac{\partial u_i}{\partial x_k}\Bigg\{nS^{(1,0,1)}_{i,\ k}(x,\hat{x},x)-R^{(1,1)}_{i,k}(x,x)R^{(0,0)}(x,\hat{x})=$$

$$
\begin{aligned}
& - R_i^{(1,0)}(x,\hat{x}) R_k^{(1,0)}(x,x) - R_k^{(1,0)}(x,\hat{x}) R_i^{(1,0)}(x,x) \\
& - \frac{n}{18B_1}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left[R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,0)}(x,\hat{x})\right] \\
& - \frac{n^2KT}{6B_1m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n} R_i^{(1,0)}(x,\hat{x})\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{n} R_k^{(1,0)}(x,\hat{x})\right)\right]\Bigg\} \\
+ \frac{2}{3n^2}\left(\frac{\partial u_k}{\partial x_k}\right)\Bigg\{ & nR^{(2,0)}(x,\hat{x}) - \Bigg[nS_{l,\ l}^{(1,0,1)}(x,\hat{x},x) \\
& - R^{(0,0)}(x,\hat{x}) R_{l,l}^{(1,1)}(x,x) - 2R_l^{(1,0)}(x,\hat{x}) R_l^{(1,0)}(x,x) \\
& - \frac{n}{9B_1}\left(\frac{\partial u_l}{\partial x_l}\right)\left(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,0)}(x,\hat{x})\right) \\
& - \frac{n^2KT}{3B_1m}\frac{\partial}{\partial x_l}\left(\frac{1}{n} R_l^{(1,0)}(x,\hat{x})\right)\Bigg]\Bigg\} \\
+ \frac{\partial}{\partial x_i}\Bigg\{ & -\frac{4}{3n^2} R_k^{(1,0)}(x,\hat{x}) R_{k,i}^{(1,1)}(x,x) - \frac{2}{3n^2} R_i^{(1,0)}(x,\hat{x}) R_{k,k}^{(1,1)}(x,x) \\
& + \frac{5}{3n^2} R_i^{(1,0)}(x,\hat{x})\left[\frac{3KT}{m} R^{(0,0)}(x,x) - R^{(0,2)}(x,x)\right] \\
& + \frac{5}{3n^2} R_i^{(1,0)}(x,x)\left[\frac{3KT}{m} R^{(0,0)}(x,\hat{x}) - R^{(2,0)}(x,\hat{x})\right] \\
& + \frac{5}{3n^2} R^{(0,0)}(x,\hat{x})\left[\frac{3KT}{m} R_i^{(1,0)}(x,x) - R_i^{(1,2)}(x,x)\right] \\
& + \frac{5}{3n}\left[S_i^{(1,0,2)}(x,\hat{x},x) - \frac{3KT}{m} S_i^{(1,0,0)}(x,\hat{x},x)\right] \\
& + \frac{2KT}{9B_1mn} R_i^{(1,0)}(x,\hat{x})\left(\frac{\partial u_k}{\partial x_k}\right) - \frac{KT}{3B_1mn} R_k^{(1,0)}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \\
& - \frac{5K}{6B_1mn}\left(R^{(2,0)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,0)}(x,\hat{x})\right)\frac{\partial T}{\partial x_i} \\
& - \frac{5KT^2}{12B_1m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT} R^{(2,0)}(x,\hat{x}) - \frac{3K}{mn} R^{(0,0)}(x,\hat{x})\right] \\
& + \frac{5KT}{m} R_i^{(1,0)}(x,\hat{x})\Bigg\} + \frac{\partial R_k^{(2,1)}}{\partial \hat{x}_k} = 0 \qquad (127)
\end{aligned}
$$

$$\frac{\mathfrak{D} R^{(2,1)}_{j}}{\mathfrak{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,1)}_{j} + 2\frac{Du_i}{Dt} R^{(1,1)}_{i,j} + \frac{\hat{D}\hat{u}_j}{\hat{D}t} R^{(2,0)}$$

$$+ \frac{2}{n^2}\frac{\partial u_i}{\partial x_k}\left\{ nS^{(1,1,1)}_{i,j,k}(x,\hat{x},x) - R^{(1,1)}_{i,k}(x,x)R^{(0,1)}_{j}(x,\hat{x}) \right.$$

$$- R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,0)}_{k}(x,x) - R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,0)}_{i}(x,x)$$

$$- \frac{n}{18B_1}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)\left[R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m} R^{(0,1)}_{j}(x,\hat{x}) \right]$$

$$\left. - \frac{n^2KT}{6B_1m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n} R^{(1,1)}_{i,j}(x,\hat{x}) \right) + \frac{\partial}{\partial x_i}\left(\frac{1}{n} R^{(1,1)}_{k,j}(x,\hat{x}) \right) \right] \right\}$$

$$+ \frac{2}{3n^2}\left(\frac{\partial u_k}{\partial x_k} \right)\left\{ n^2R^{(2,1)}_{j}(x,\hat{x}) - \left[nS^{(1,1,1)}_{l,j,l}(x,\hat{x},x) \right. \right.$$

$$- R^{(0,1)}_{j}(x,\hat{x})R^{(1,1)}_{l,l}(x,x) - 2R^{(1,1)}_{l,j}(x,\hat{x})R^{(1,0)}_{l}(x,x)$$

$$- \frac{n}{9B_1}\left(\frac{\partial u_l}{\partial x_l} \right)\left(R^{(2,1)}_{j}(x,\hat{x}) - \frac{3KT}{m} R^{(0,1)}_{j}(x,\hat{x}) \right)$$

$$\left. \left. - \frac{n^2KT}{3B_1m}\frac{\partial}{\partial x_l}\left(\frac{1}{n} R^{(1,1)}_{l,j}(x,\hat{x}) \right) \right] \right\}$$

$$+ R^{(2,1)}_{k}\frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial}{\partial x_i}\left\{ -\frac{4}{3n^2} R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,1)}_{k,i}(x,x) \right.$$

$$- \frac{2}{3n^2} R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,1)}_{k,k}(x,x) + \frac{5}{3n^2} R^{(1,1)}_{i,j}(x,\hat{x})$$

$$\times \left[\frac{3KT}{m} R^{(0,0)}(x,x) - R^{(0,2)}(x,x) \right]$$

$$+ \frac{5}{3n^2} R^{(1,0)}_{i}(x,x)\left[\frac{3KT}{m} R^{(0,1)}_{j}(x,\hat{x}) - R^{(2,1)}_{j}(x,\hat{x}) \right]$$

$$+ \frac{5}{3n^2} R^{(0,1)}_{j}(x,\hat{x})\left[\frac{3KT}{m} R^{(1,0)}_{i}(x,x) - R^{(1,2)}_{i}(x,x) \right]$$

$$+ \frac{5}{3n}\left[S^{(1,1,2)}_{i,j}(x,\hat{x},x) - \frac{3KT}{m} S^{(1,1,0)}_{i,j}(x,\hat{x},x) \right] =$$

$$+\frac{2KT}{9B_1mn}R^{(1,1)}_{i,j}(x,\hat{x})\left(\frac{\partial u_k}{\partial x_k}\right)-\frac{KT}{3B_1mn}R^{(1,1)}_{k,j}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)$$

$$-\frac{5K}{6B_1mn}\left(R^{(2,1)}_{j}(x,\hat{x})-\frac{3KT}{m}R^{(0,1)}_{j}(x,\hat{x})\right)\frac{\partial T}{\partial x_i}$$

$$-\frac{5KT^2}{12B_1m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT}R^{(2,1)}_{j}(x,\hat{x})-\frac{3K}{mn}R^{(0,1)}_{j}(x,\hat{x})\right]$$

$$\left.+\frac{5KT}{m}R^{(1,1)}_{i,j}(x,\hat{x})\right\}$$

$$+\frac{\partial}{\partial\hat{x}_k}\left\{\frac{1}{\hat{n}}S^{(2,1,1)}_{j,k}(x,\hat{x},\hat{x})-\frac{1}{\hat{n}^2}R^{(1,1)}_{j,k}(\hat{x},\hat{x})R^{(2,0)}(x,\hat{x})\right.$$

$$-\frac{1}{\hat{n}^2}R^{(2,1)}_{j}(x,\hat{x})R^{(1,0)}_{k}(\hat{x},\hat{x})-\frac{1}{\hat{n}^2}R^{(2,1)}_{k}(x,\hat{x})R^{(1,0)}_{j}(\hat{x},\hat{x})$$

$$-\frac{1}{18B_1\hat{n}}\left(\frac{\partial\hat{u}_j}{\partial\hat{x}_k}+\frac{\partial\hat{u}_k}{\partial\hat{x}_j}\right)\left[R^{(2,2)}(x,\hat{x})-\frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\right]$$

$$\left.-\frac{KT}{6B_1m}\left[\frac{\partial}{\partial\hat{x}_k}\left(\frac{1}{\hat{n}}R^{(2,1)}_{j}(x,\hat{x})\right)+\frac{\partial}{\partial\hat{x}_j}\left(\frac{1}{\hat{n}}R^{(2,1)}_{k}(x,\hat{x})\right)\right]\right\}$$

$$+\frac{1}{3}\frac{\partial}{\partial\hat{x}_j}\left\{R^{(2,2)}(x,\hat{x})-\left[\frac{1}{\hat{n}}S^{(2,1,1)}_{l,l}(x,\hat{x},\hat{x})\right.\right.$$

$$-\frac{1}{\hat{n}^2}R^{(2,0)}(x,\hat{x})R^{(1,1)}_{l,l}(\hat{x},\hat{x})-\frac{2}{\hat{n}^2}R^{(2,1)}_{l}(x,\hat{x})R^{(1,0)}_{l}(\hat{x},\hat{x})$$

$$-\frac{1}{9B_1\hat{n}}\left(\frac{\partial\hat{u}_l}{\partial\hat{x}_l}\right)\left(R^{(2,2)}(x,\hat{x})-\frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\right)$$

$$\left.\left.-\frac{n^2K\hat{T}}{3B_1m}\frac{\partial}{\partial\hat{x}_l}\left(\frac{1}{\hat{n}}R^{(2,1)}_{l}(x,\hat{x})\right)\right]\right\}=0\qquad(1\leqslant j\leqslant 3)\tag{128}$$

$$\frac{\mathfrak{D}R^{(2,2)}}{\mathfrak{D}t}+\left(\frac{\partial u_k}{\partial x_k}+\frac{\partial\hat{u}_k}{\partial\hat{x}_k}\right)R^{(2,2)}+2\frac{Du_i}{Dt}R^{(1,2)}_{i}+2\frac{\hat{D}\hat{u}_j}{\hat{D}t}R^{(2,1)}_{j}$$

$$+\frac{2}{n^2}\frac{\partial u_i}{\partial x_k}\left\{nS^{(1,2,1)}_{i,\ ,k}(x,\hat{x},x)-R^{(1,1)}_{i,k}(x,x)R^{(0,2)}(x,\hat{x})\right.$$

$$-R^{(1,2)}_{i}(x,\hat{x})R^{(1,0)}_{k}(x,x)-R^{(1,2)}_{k}(x,\hat{x})R^{(1,0)}_{i}(x,x)=$$

$$- \frac{n}{18B_1}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\left[R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,2)}(x,\hat{x})\right]$$

$$- \frac{n^2KT}{6B_1 m}\left[\frac{\partial}{\partial x_k}\left(\frac{1}{n}R^{(1,2)}_{i}(x,\hat{x})\right) + \frac{\partial}{\partial x_i}\left(\frac{1}{n}R^{(1,2)}_{k}(x,\hat{x})\right)\right]\Bigg\}$$

$$+ \frac{2}{3n^2}\left(\frac{\partial u_k}{\partial x_k}\right)\Bigg\{n^2R^{(2,2)}(x,\hat{x}) - \Bigg[nS^{(1,2,1)}_{l,\;l}(x,\hat{x},x) - R^{(0,2)}(x,\hat{x})R^{(1,1)}_{l,l}(x,x)$$

$$- 2R^{(1,2)}_{l}(x,\hat{x})R^{(1,0)}_{l}(x,x)$$

$$- \frac{n}{9B_1}\left(\frac{\partial u_l}{\partial x_l}\right)\left(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m}R^{(0,2)}(x,\hat{x})\right)$$

$$- \frac{n^2KT}{3B_1 m}\frac{\partial}{\partial x_l}\left(\frac{1}{n}R^{(1,2)}_{l}(x,\hat{x})\right)\Bigg]\Bigg\}$$

$$+ \frac{2}{\hat{n}^2}\frac{\partial \hat{u}_j}{\partial \hat{x}_k}\Bigg\{\hat{n}S^{(2,1,1)}_{j,k}(x,\hat{x},\hat{x}) - R^{(1,1)}_{j,k}(\hat{x},\hat{x})R^{(2,0)}(x,\hat{x})$$

$$- R^{(2,1)}_{j}(x,\hat{x})R^{(1,0)}_{k}(\hat{x},\hat{x}) - R^{(2,1)}_{k}(x,\hat{x})R^{(1,0)}_{j}(\hat{x},\hat{x})$$

$$- \frac{\hat{n}}{18B_1}\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_j}\right)\left[R^{(2,2)}(x,\hat{x}) - \frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\right]$$

$$- \frac{\hat{n}^2K\hat{T}}{6B_1 m}\left[\frac{\partial}{\partial \hat{x}_k}\left(\frac{1}{\hat{n}}R^{(2,1)}_{j}(x,\hat{x})\right) + \frac{\partial}{\partial \hat{x}_j}\left(\frac{1}{\hat{n}}R^{(2,1)}_{k}(x,\hat{x})\right)\right]\Bigg\}$$

$$+ \frac{2}{3\hat{n}^2}\left(\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right)\Bigg\{\hat{n}^2R^{(2,2)}(x,\hat{x}) - \Bigg[\hat{n}S^{(2,1,1)}_{l,l}(x,\hat{x},\hat{x}) - R^{(2,0)}(x,\hat{x})R^{(1,1)}_{l,l}(\hat{x},\hat{x})$$

$$- 2R^{(2,1)}_{l}(x,\hat{x})R^{(1,0)}_{l}(\hat{x},\hat{x}) - \frac{\hat{n}}{9B_1}\left(\frac{\partial \hat{u}_l}{\partial \hat{x}_l}\right)\Bigg(R^{(2,2)}(x,\hat{x})$$

$$- \frac{3K\hat{T}}{m}R^{(2,0)}(x,\hat{x})\Bigg) - \frac{\hat{n}^2K\hat{T}}{3B_1 m}\frac{\partial}{\partial \hat{x}_l}\left(\frac{1}{\hat{n}}R^{(2,1)}_{l}(x,\hat{x})\right)\Bigg]\Bigg\}$$

$$+ \frac{\partial}{\partial x_i}\Bigg\{- \frac{4}{3n^2}R^{(1,2)}_{k}(x,\hat{x})R^{(1,1)}_{k,i}(x,x) - \frac{2}{3n^2}R^{(1,2)}_{i}(x,\hat{x})R^{(1,1)}_{k,k}(x,x) =$$

$$
\begin{aligned}
&+ \frac{5}{3n^2} R_i^{(1,2)}(x,\hat{x})\left[\frac{3KT}{m} R^{(0,0)}(x,x) - R^{(0,2)}(x,x)\right] \\
&+ \frac{5}{3n^2} R_i^{(1,0)}(x,x)\left[\frac{3KT}{m} R^{(0,2)}(x,\hat{x}) - R^{(2,2)}(x,\hat{x})\right] \\
&+ \frac{5}{3n^2} R^{(0,2)}(x,\hat{x})\left[\frac{3KT}{m} R_i^{(1,0)}(x,x) - R_i^{(1,2)}(x,x)\right] \\
&+ \frac{5}{3n}\left[S_i^{(1,2,2)}(x,\hat{x},x) - \frac{3KT}{m} S_i^{(1,2,0)}(x,\hat{x},x)\right] \\
&+ \frac{2KT}{9B_1 mn} R_i^{(1,2)}(x,\hat{x})\left(\frac{\partial u_k}{\partial x_k}\right) - \frac{KT}{3B_1 mn} R_k^{(1,2)}(x,\hat{x})\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \\
&- \frac{5K}{6B_1 mn}\left(R^{(2,2)}(x,\hat{x}) - \frac{3KT}{m} R^{(0,2)}(x,\hat{x})\right)\frac{\partial T}{\partial x_i} \\
&- \frac{5KT^2}{12B_1 m}\frac{\partial}{\partial x_i}\left[\frac{1}{nT} R^{(2,2)}(x,\hat{x}) - \frac{3K}{mn} R^{(0,2)}(x,\hat{x})\right] + \frac{5KT}{m} R_i^{(1,2)}(x,\hat{x})\Bigg\} \\
+ \frac{\partial}{\partial \hat{x}_i}\Bigg\{ &- \frac{4}{3\hat{n}^2} R_k^{(2,1)}(x,\hat{x}) R_{i,k}^{(1,1)}(x,\hat{x}) - \frac{2}{3\hat{n}^2} R_i^{(2,1)}(x,\hat{x}) R_{k,k}^{(1,1)}(\hat{x},\hat{x}) \\
&+ \frac{5}{3\hat{n}^2} R_i^{(2,1)}(x,\hat{x})\left[\frac{3K\hat{T}}{m} R^{(0,0)}(\hat{x},\hat{x}) - R^{(2,0)}(\hat{x},\hat{x})\right] \\
&+ \frac{5}{3\hat{n}^2} R_i^{(0,1)}(\hat{x},\hat{x})\left[\frac{3K\hat{T}}{m} R^{(2,0)}(x,\hat{x}) - R^{(2,2)}(x,\hat{x})\right] \\
&+ \frac{5}{3\hat{n}^2} R^{(2,0)}(x,\hat{x})\left[\frac{3K\hat{T}}{m} R_i^{(0,1)}(\hat{x},\hat{x}) - R_i^{(2,1)}(\hat{x},\hat{x})\right] \\
&+ \frac{5}{3\hat{n}}\left[S_i^{(2,1,2)}(x,\hat{x},\hat{x}) - \frac{3K\hat{T}}{m} S_i^{(2,0,1)}(x,\hat{x},\hat{x})\right] \\
&+ \frac{2K\hat{T}}{9B_1 m\hat{n}} R_i^{(2,1)}(x,\hat{x})\left(\frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) - \frac{K\hat{T}}{3B_1 m\hat{n}} R_k^{(2,1)}(x,\hat{x})\left(\frac{\partial \hat{u}_i}{\partial \hat{x}_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_i}\right) \\
&- \frac{5K}{6B_1 m\hat{n}}\left(R^{(2,2)}(x,\hat{x}) - \frac{3K\hat{T}}{m} R^{(2,0)}(x,\hat{x})\right)\frac{\partial \hat{T}}{\partial \hat{x}_i} \\
&- \frac{5K\hat{T}^2}{12B_1 m}\frac{\partial}{\partial \hat{x}_i}\left[\frac{1}{\hat{n}\hat{T}} R^{(2,2)}(x,\hat{x}) - \frac{3K}{m\hat{n}} R^{(2,0)}(x,\hat{x})\right] \\
&+ \frac{5K\hat{T}}{m} R_i^{(2,1)}(x,\hat{x})\Bigg\} = 0 \qquad (129)
\end{aligned}
$$

Equations (53)–(55), together with (114)–(129), represent 155 equations which form the system of approximate equations of order $1\frac{2}{3}$ governing the motion of turbulent flows. The system of the equations will be greatly simplified if the flow is supposed to be incompressible:

$$\frac{\partial S^{(1,1,1)}_{i,j,k}}{\partial \check{x}_k} = 0 \qquad (1 \leqslant i, j \leqslant 3)$$

$$\frac{\partial S^{(2,1,1)}_{j,k}}{\partial \check{x}_k} = 0 \qquad (1 \leqslant j \leqslant 3)$$

$$\begin{aligned}
&\frac{\Delta S^{(1,1,1)}_{i,j,k}}{\Delta t} + \frac{\partial u_i}{\partial x_k} S^{(1,1,1)}_{k,j,l} + \frac{\partial \hat{u}_j}{\partial \hat{x}_k} S^{(1,1,1)}_{i,k,l} + \frac{\partial \check{u}_l}{\partial \check{x}_k} S^{(1,1,1)}_{i,j,k} \\
&\quad + \frac{\partial}{\partial x_k}\left\{\frac{1}{n}\left[R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,1)}_{k,l}(x,\check{x}) + R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,1)}_{i,l}(x,\check{x})\right]\right\} \\
&\quad + \frac{1}{3}\frac{\partial}{\partial x_i}\left[S^{(2,1,1)}_{j,l} - \frac{2}{n}R^{(1,1)}_{k,j}(x,\hat{x})R^{(1,1)}_{k,l}(x,\check{x})\right] \\
&\quad + \frac{\partial}{\partial \hat{x}_k}\left\{\frac{1}{\hat{n}}\left[R^{(1,1)}_{i,k}(x,\hat{x})R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(1,1)}_{i,j}(x,\hat{x})R^{(1,1)}_{k,l}(\hat{x},\check{x})\right]\right\} \\
&\quad + \frac{1}{3}\frac{\partial}{\partial \hat{x}_j}\left\{S^{(1,2,1)}_{i,\ l} - \frac{2}{\hat{n}}R^{(1,1)}_{i,k}(x,\hat{x})R^{(1,1)}_{k,l}(\hat{x},\check{x})\right] \\
&\quad + \frac{\partial}{\partial \check{x}_k}\left\{\frac{1}{\check{n}}\left[R^{(1,1)}_{i,k}(x,\check{x})R^{(1,1)}_{j,l}(\hat{x},\check{x}) + R^{(1,1)}_{i,l}(x,\check{x})R^{(1,1)}_{j,k}(\hat{x},\check{x})\right]\right\} \\
&\quad + \frac{1}{3}\frac{\partial}{\partial \check{x}_l}\left[S^{(1,1,2)}_{i,j} - \frac{2}{\check{n}}R^{(1,1)}_{i,k}(x,\check{x})R^{(1,1)}_{j,k}(\hat{x},\check{x})\right] = 0 \qquad (1 \leqslant i, j, l \leqslant 3)
\end{aligned}$$

$$\frac{\partial R^{(1,1)}_{i,k}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant i \leqslant 3)$$

$$\frac{\partial R^{(2,1)}_{k}}{\partial \hat{x}_k} = 0$$

$$\begin{aligned}
&\frac{\mathcal{D} R^{(1,1)}_{i,j}}{\mathcal{D} t} + R^{(1,1)}_{k,j}\frac{\partial u_i}{\partial x_k} + R^{(1,1)}_{i,k}\frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{1}{n}\frac{\partial}{\partial x_k}\left[S^{(1,1,1)}_{i,j,k}(x,\hat{x},x)\right] \\
&\quad - \kappa\frac{\partial}{\partial x_k}\left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)R^{(2,1)}_{j}(x,\hat{x})\right] - \nu\Delta R^{(1,1)}_{i,j}(x,\hat{x}) \\
&\quad + \frac{1}{3}\frac{\partial}{\partial x_i}R^{(2,1)}_{j}(x,\hat{x}) - \frac{1}{3n}\frac{\partial}{\partial x_i}\left[S^{(1,1,1)}_{l,j,l}(x,\hat{x},x)\right] \\
&\quad + \frac{1}{n}\frac{\partial}{\partial \hat{x}_k}\left[S^{(1,1,1)}_{i,j,k}(x,\hat{x},\hat{x})\right] =
\end{aligned}$$

$$
\begin{aligned}
&-\kappa\frac{\partial}{\partial \hat{x}_k}\left[\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_j}\right)R^{(1,2)}_{i}(x,\hat{x})\right]\\
&-\nu\hat{\Delta}R^{(1,1)}_{i,j}(x,\hat{x})+\frac{1}{3}\frac{\partial}{\partial \hat{x}_j}R^{(1,2)}_{i}(x,\hat{x})\\
&-\frac{1}{3n}\frac{\partial}{\partial \hat{x}_j}\left[S^{(1,1,1)}_{i,l,l}(x,\hat{x},\hat{x})\right]=0\\
&(1\leqslant i,j\leqslant 3)
\end{aligned}
$$

where $\nu=\mu/\rho$ and $\kappa=1/18nB_1$. The constant κ is a new transport coefficient. The existence of the transport coefficient κ means that the deformation of the fluid produces not only the stress, but also the stress correlation. This fact is a consequence of the Enskog–Chapman expansion.

15. A SUFFICIENT CONDITION FOR A FLOW BEING LAMINAR

If we neglect the three-point correlations S's, the last equation in the last section will take the following form:

$$
\begin{aligned}
&\frac{\partial R^{(1,1)}_{i,j}}{\partial t}+u_k\frac{\partial R^{(1,1)}_{i,j}}{\partial x_k}+\hat{u}_k\frac{\partial R^{(1,1)}_{i,j}}{\partial \hat{x}_k}+R^{(1,1)}_{k,j}\frac{\partial u_i}{\partial x_k}+R^{(1,1)}_{i,k}\frac{\partial \hat{u}_j}{\partial \hat{x}_k}\\
&-\kappa\frac{\partial}{\partial x_k}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)R^{(2,1)}_{j}(x,\hat{x})\right]\\
&-\nu\Delta R^{(1,1)}_{i,j}(x,\hat{x})+\frac{1}{3}\frac{\partial}{\partial x_i}R^{(2,1)}_{j}(x,\hat{x})\\
&-\kappa\frac{\partial}{\partial \hat{x}_k}\left[\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_j}\right)R^{(1,2)}_{i}(x,\hat{x})\right]-\nu\hat{\Delta}R^{(1,1)}_{i,j}(x,\hat{x})\\
&+\frac{1}{3}\frac{\partial}{\partial \hat{x}_j}R^{(1,2)}_{i}(x,\hat{x})=0\qquad(1\leqslant i,j\leqslant 3)
\end{aligned}
\tag{130}
$$

In the classical theory, a flow is a solution of the Navier–Stokes equations, and a laminar flow is a stable flow. In our scheme, a flow is a solution of the system of partial differential equations which govern not only the evolution of the density and the velocity of the flow, but also that of the correlations, and a laminar flow is a flow with vanishing correlations $R^{(1,1)}_{i,j}$. Now we are going to get a sufficient condition for a flow being laminar in our scheme.

From (130) we have

$$\frac{\partial u_k}{\partial x_i}\frac{\partial R^{(1,1)}_{i,j}}{\partial x_k}+\frac{\partial R^{(1,1)}_{k,j}}{\partial x_i}\frac{\partial u_i}{\partial x_k}-\kappa\frac{\partial^2}{\partial x_i\,\partial x_k}\left[\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)R^{(2,1)}_{j}\right]$$
$$+\frac{1}{3}\Delta R^{(2,1)}_{j}=0$$

$$-\frac{\partial}{\partial x_i}\left\{\left[\frac{1}{3}\delta_{ik}-\kappa\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\right]\frac{\partial R^{(2,1)}_{j}}{\partial x_k}\right\}+\kappa\frac{\partial}{\partial x_i}\left(\Delta u_i R^{(2,1)}_{j}\right)$$
$$=2\frac{\partial u_i}{\partial x_k}\frac{\partial R^{(1,1)}_{k,j}}{\partial x_i}$$

Now we make the following assumption: All the correlations take the null boundary value. Having made this assumption, we have

$$\int\left[\frac{1}{3}\delta_{ik}-\kappa\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)\right]\frac{\partial R^{(2,1)}_{j}}{\partial x_i}\frac{R^{(2,1)}_{j}}{\partial x_k}\,dx\,d\hat{x}$$
$$=2\int R^{(2,1)}_{j}\frac{\partial u_i}{\partial x_k}\frac{\partial R^{(1,1)}_{k,j}}{\partial x_i}\,dx\,d\hat{x}$$

In deriving the above equality we have taken the following fact, which is easy to verify, into consideration:

$$\int R^{(2,1)}_{j}\frac{\partial}{\partial x_i}\left[\Delta u_i R^{(2,1)}_{j}\right]dx\,d\hat{x}=0$$

Let $\alpha \geqslant \beta \geqslant \gamma$ be the eigenvalues of the stretching tensor $(\partial u_i/\partial x_k + \partial u_k/\partial x_i)$. Assuming that

$$\tfrac{1}{3} > \kappa\alpha \tag{131}$$

we have

$$\int\left[\frac{\partial R^{(2,1)}_{j}}{\partial x_i}\right]^2 dx\,d\hat{x}\leqslant\frac{2}{\frac{1}{3}-\kappa\alpha}\left|\int\frac{\partial R^{(2,1)}_{j}}{\partial x_k}\frac{\partial u_i}{\partial x_k}R^{(1,1)}_{k,j}\,dx\,d\hat{x}\right|$$
$$\leqslant\frac{6M}{1-3\kappa\alpha}\left[\int\left(\frac{\partial R^{(2,1)}_{j}}{\partial x_i}\right)^2 dx\,d\hat{x}\right]^{1/2}\left[\int\left(R^{(1,1)}_{i,j}\right)^2 dx\,d\hat{x}\right]^{1/2}$$

where M is the norm of the matrix $(\partial u_i/\partial x_k)$ in the three-dimensional

Euclidean space. Hence,

$$\left[\int\left(\frac{\partial R_{j}^{(2,1)}}{\partial x_i}\right)^2 dx\,d\hat{x}\right]^{1/2} \leqslant \frac{6M}{1-3\kappa\alpha}\left[\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}\right]^{1/2} \tag{132}$$

On the other hand, we get the following equality from equality (130):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x} + \frac{1}{2}\int R_{i,j}^{(1,1)}R_{k,j}^{(1,1)}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)dx\,d\hat{x}\\
&+\frac{1}{2}\int R_{i,k}^{(1,1)}R_{i,j}^{(1,1)}\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_j}\right)dx\,d\hat{x}\\
&+\kappa\int\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)R_{j}^{(2,1)}\,dx\,d\hat{x}\\
&+\kappa\int\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k}+\frac{\partial \hat{u}_k}{\partial \hat{x}_j}\right)R_{i}^{(1,2)}\,dx\,d\hat{x}\\
&+\nu\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}=0
\end{aligned} \tag{133}$$

In virtue of the well-known inequality[3]

$$A\left\{\int\left[\left(\frac{\partial f}{\partial x_k}\right)^2+\left(\frac{\partial f}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}\right\}^{1/2} \geqslant \left(\int |f|^2\,dx\,d\hat{x}\right)^{1/2} \tag{134}$$

We deduce the following inequality from (133):

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x} \leqslant &-\nu\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}\\
&+\max(\alpha,-\gamma)\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}\\
&+\kappa(\alpha^2+\beta^2+\gamma^2)^{1/2}\left\{\left[\int\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2 dx\,d\hat{x}\right]^{1/2}\left[\int\left(R_{j}^{(2,1)}\right)^2 dx\,d\hat{x}\right]^{1/2}\right. =
\end{aligned}$$

$$+\left[\int\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2 dx\,d\hat{x}\right]^{1/2}\left[\int\left(R_i^{(1,2)}\right)^2 dx\,d\hat{x}\right]^{1/2}\Bigg\}$$

$$\leqslant -\nu\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}$$

$$+\max(\alpha,-\gamma)\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}$$

$$+\kappa(\alpha^2+\beta^2+\gamma^2)^{1/2}\left\{\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}\right\}^{1/2}$$

$$\times\frac{24AM}{1-3\kappa\alpha}\left[\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}\right]^{1/2}$$

$$\leqslant -\nu\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}$$

$$+\max(\alpha,-\gamma)\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}$$

$$+\frac{24\kappa AM(\alpha^2+\beta^2+\gamma^2)^{1/2}}{1-3\kappa\alpha}$$

$$\cdot\frac{1}{2}\left\{A^{1/2}\int\left[\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial x_k}\right)^2+\left(\frac{\partial R_{i,j}^{(1,1)}}{\partial \hat{x}_k}\right)^2\right]dx\,d\hat{x}\right.$$

$$\left.+\frac{1}{A^{1/2}}\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}\right\}$$

$$\leqslant\left[-\frac{\nu}{A}+\frac{24\kappa A^{1/2}M(\alpha^2+\beta^2+\gamma^2)^{1/2}}{1-3\kappa\alpha}+\max(\alpha,-\gamma)\right]$$

$$\times\int\left(R_{i,j}^{(1,1)}\right)^2 dx\,d\hat{x}$$

if

$$\nu \geqslant \frac{24\kappa(A)^{3/2}M(\alpha^2+\beta^2+\gamma^2)^{1/2}}{1-3\kappa\alpha}$$

Now it is easy to see that a sufficient condition for the flow being laminar flow is the inequality

$$\nu \geqslant \frac{24\kappa A^{3/2}M(\alpha^2+\beta^2+\gamma^2)^{1/2}}{1-3\kappa\alpha} + A\max(\alpha, -\gamma)$$

combined with inequality (131). It is worth while mentioning that if κ were equal to zero, the conclusions in our theory and in the classical stability theory[3] would be identical. This fact is very interesting, because the starting points of the two theories are quite different. We think the new transport coefficient κ will play an important role in the further study of turbulence phenomena.

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